Duality relationships and supermultiplet symmetry in the $O(8)$ pair-coupling model

2007 J. Phys. A: Math. Theor. 40 471

(http://iopscience.iop.org/1751-8121/40/3/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 471–500 doi:10.1088/1751-8113/40/3/009

Duality relationships and supermultiplet symmetry in the $O(8)$ pair-coupling model

D J Rowe[1] **and M J Carvalho**[2]

[1] Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
[2] Department of Physics, Ryerson University, Toronto, Ontario M5B 2K3, Canada

Received 24 August 2006
Published 20 December 2006
Online at stacks.iop.org/JPhysA/40/471

Abstract
A long suspected duality relationship is established between representations of an $O(8)$ group, whose infinitesimal generators include ($L = 0$, $S = 0$, $T = 1$) and ($L = 0$, $S = 1$, $T = 0$), two-nucleon pair operators, and those of the group of orthogonal transformations of the spatial wavefunctions of the nucleons. The implications of this duality in the classification of nuclear states in LST coupling are explored for the 2s1d shell.

PACS numbers: 02.20.−a, 21.60.−n, 21.60.Fw

1. Introduction

Schematic interactions, which lead to solvable models, have had an enormous impact on physics. The so-called *pairing forces* have been particularly influential in explaining superconducting systems in terms of the tendency of fermions to form spin-zero *Cooper pairs* as a result of the mutual attractive forces between them; cf, for example, Talmi's book [1] for an introduction to the numerous applications of pair-coupling and related seniority coupling schemes in nuclear physics. One reason for the remarkable success of the pair-coupling models is that they not only describe the essential physics of the pairing phenomena, but they are also computationally tractable algebraic models with Hamiltonians that are expressible in terms of the Lie algebras of corresponding dynamical groups. A second and equally important reason is that a dynamical group for such a pair-coupling model is a direct product of two groups whose representations define each other uniquely in a remarkable duality relationship. As a result, both representations of the dual pair can be exploited to give complementary information in the description of a model with pairing interactions [2]. In practice, this means that the irreducible subspaces for such a model are tensor products of dual subspaces with a pairing Hamiltonian acting in one subspace and a complementary Hamiltonian acting in the other.

Many such duality relationships are now known. The first, and best known, is the famous Schur–Weyl duality between the representations of the symmetric and unitary groups. Schur–Weyl duality underlies the whole representation theory of the symmetric and unitary groups and leads to many invaluable results, e.g., for the branching rules of many subgroups of the unitary groups [2]. Several of these results will be used in this paper. A second duality relationship between $SU(2)$ quasispin and the compact symplectic groups, implicit in the nuclear and atomic shell-model couplings schemes of Racah [3] and Flowers [4], was uncovered by Kerman [5] and Helmers [6], in the context of nuclear pair-coupling models.

The paper of Helmers, although virtually unknown outside of nuclear physics, is remarkable because it identifies the conditions for what we now describe as *dual pairs of group representations* (defined in the following section). Helmers showed that the infinitesimal generators of the symplectic group used by Flowers could be regarded as skew-symmetric bilinear forms that are the invariants of another symplectic group, which he called the *commutator group* of Flower's symplectic group. He then went on to show that the Casimir invariants of these two mutually commuting symplectic groups are linearly related. Moreover, by consideration of the characters of their direct product, he showed that *the irreducible representations of the two symplectic groups that are coupled together are in one-to-one correspondence and have complementary Young tableaux*.

In the light of these discoveries, Flowers and Szpikowski [7] went on to show that the diagonalization of a charge-independent pairing Hamiltonian leads to a classification of the states of a system of neutrons and protons in jj-coupling in terms of the seniority and reduced isotopic spin quantum numbers introduced in their earlier paper [4]. Thus, the $SU(2) \sim USp(2)$ quasispin pair-coupling model for a system of either neutrons or protons was extended to an isospin-invariant model for nucleons with an $SO(5) \sim USp(4)$ dynamical group, where USp denotes a compact unitary-symplectic group.

Shortly afterwards, Flowers and Szpikowski [8] proposed an extension to the pair-coupling model. Recognizing that the $SO(5)$ model includes interactions only in $J = 0, T = 1$ pair states, they proposed an LST version of the model which admits interactions in both $S = 1, T = 0$ and $S = 0, T = 1$ pair-coupled $L = 0$ states. They also showed that this model has a spectrum generating algebra given by the Lie algebra of the group $SO(8)$ which contains Wigner's $U(4)$ supermultiplet group [9] as a subgroup. Moreover, they were able to express their Hamiltonian in terms of the Casimir invariants of $SO(8)$ and $SU(4)$ and thereby derive its spectrum. As it happened, an equivalent result had been derived earlier by Bayman [10] by observing that the $L = 0$ pairing interaction is an invariant of the group $SO(2l + 1)$. In fact, it transpired that while Flowers and Szpikowski had discovered a *dynamical group* for the $L = 0$ pairing Hamiltonian, Bayman had discovered a complementary *symmetry group*. However, a possible duality relationship between the two group structures was not investigated at that time.

A third duality relationship was observed (again in nuclear physics) by Moshinsky and Quesne [11] who described the relationship as *complementarity*. Based on results obtained by Chaćon [12], Moshinsky and Quesne observed that the representations of the non-compact symplectic groups $Sp(n, \mathbb{R})$ and certain orthogonal groups, that occur in the decomposition of many-particle harmonic-oscillator spaces, are dual to one another. Elegant (albeit difficult) proofs of a more general duality theorem on multi-dimensional harmonic-oscillator spaces were subsequently given by Kashiwara and Vergne [13] and by Howe [14] using the sophisticated mathematics of invariant theory. As a result, the duality concept has become widely known in mathematics in terms of Howe's so-called *dual reductive pairs*. A useful review of this subject from the mathematics perspective has been given by Howe in [15].

Our purpose in this paper is to establish, and explore the implications of, an orthogonal–orthogonal duality relationship based on the $L = 0$ Flowers–Szpikowsky model [8]. As emphasized by Evans [16] and Dussel *et al* [17], this model, while more complicated than the standard $J = 0$ pairing model, is often more useful and more realistic, because it allows pairing in the spatial degrees of freedom regardless of the spin and isospin of the pair. Several papers [16–22] have explored the spectra and properties of the Flowers–Szpikowsky model in terms of its $SO(8)$ algebraic structure. For example, Pang [19] gave expressions for matrix elements of the $SO(8)$ Lie algebra in irreps of seniority zero and one. These expressions were used by Evans *et al* [16] to compute the spectra for a 12-particle seniority-zero irrep for different $S = 0, T = 1$ and $S = 1, T = 0$ pairing strengths. A more general Hamiltonian, quadratic in the $SO(8)$ algebra, was used by Engel *et al* [20] to study Gamow–Teller strengths and double β decay.

Of particular relevance to the present investigation is the paper of Kota and Alcarás [22] which focuses on chains of Lie algebras which pairwise commute with those of $SO(8)$; the Kota–Alcarás paper appeared recently (during the course of this investigation) and we are not aware of any earlier publication that goes as far. Kota and Alcarás observed, for example, that the subset of $U(N)$ one-body operators that commute with the operators of the $SO(8)$ Lie algebra, in the space of any number of nucleons in $N = \sum_i (2l_i + 1)$ single-particle spatial states, is the $SO(N)$ Lie algebra. Moreover, they showed that the values of the Casimir operators for the irreps of these two algebras, that occur in combination, are linearly related, as were those of the symplectic algebras considered by Helmers. This goes a long way towards establishing the duality of the $O(8)$ and $O(N)$ groups which share their Lie algebras with those of $SO(8)$ and $SO(N)$.

It is shown in the following section that the groups $O(8)$ and $O(N)$ do indeed have dual representations on the space of many fermions occupying N spatial single-particle states. It should be emphasized, however, that to show this it is not sufficient to consider only the Lie algebras of the groups in question. This is important because, while the group $O(N)$ and its $SO(N) \subset O(N)$ subgroup share a common Lie algebra, they are distinct groups. Moreover, for even values of N, an $O(N)$ irrep may restrict to a sum of two $SO(N)$ irreps. (Recall that $SO(N)$ is the subgroup of elements of $O(N)$ with unit determinant whereas, in a general $O(N)$ irrep, $\det(g)$ can take the value ± 1. Thus, $O(N)$ irreps are distinguished by an additional $O(1)$ quantum number that is not required for $SO(N)$.)

An identification of pairs of groups with dual representations, one group acting on the spin–isospin states and the other on the spatial states of many nucleons, is of considerable value in solving shell-model problems. For example, the dual representations of the groups $O(8)$ and $O(N)$ imply that if $H = H_1 + H_2$ is a Hamiltonian with H_1 a polynomial in the $O(8)$ Lie algebra and H_2 a polynomial in the $O(N)$ Lie algebra, then $O(8)$ is a dynamical group for H_1 and a symmetry group for H_2 and, conversely, $O(N)$ is a dynamical group for H_2 and a symmetry group for H_1. Subgroup chains of groups with dual representations are even more useful. For the current model, we have the subgroup chains

$$O(8) \supset U(4) \supset SU(2)_S \times SU(2)_T, \tag{1}$$

$$U(N) \supset O(N) \supset SO(3)_L. \tag{2}$$

The remarkable fact is that because of the duality relationships between $O(8)$ and $O(N)$ and between $U(4)$ and $U(N)$ it is possible to construct basis states for the Fock space $\mathbb{F}$ that simultaneously reduce both subgroup chains, in spite of the fact that the groups $O(8)$ and $U(N)$ do not commute with one another. Such related subgroup chains are conveniently described as *dual subgroup chains*.

2. Dual pairs of group representations

Definition 1. *Two groups G_1 and G_2 are said to have dual representations on a space $\mathbb{F}$ if the following conditions hold: (i) the actions of the two groups commute with one another;*

(ii) the representation of the direct product group $G_1 \times G_2$ on $\mathbb{F}$ is fully reducible and its decomposition is multiplicity free; (iii) a particular irrep of G_1 only occurs in combination with a single uniquely defined irrep of G_2 and vice versa.

In the following, we consider $\mathbb{F}$ to be a many-nucleon Fock space for which the nucleon creation and annihilation operators are indexed by a double set of labels $\{a^\dagger_{\sigma i}, a^{\sigma i}\}$, where $i = 1, \dots, N$ indexes N spatial indices and σ indexes the four spin–isopin states of a nucleon. These operators obey the standard fermion anti-commutation relations

$$\{a^{\sigma i}, a^\dagger_{\tau j}\} = \delta^\sigma_\tau \delta^i_j, \qquad \{a^\dagger_{\sigma i}, a^\dagger_{\tau j}\} = \{a^{\sigma i}, a^{\tau j}\} = 0. \tag{3}$$

In this section, we first review the well-known unitary–unitary duality on this space and then derive an orthogonal–orthogonal duality; we are not aware that the latter has previously been substantiated.

2.1. Unitary–unitary duality

The Fock space $\mathbb{F}$ carries a reducible representation of a direct product group, $U(4) \times U(N)$, whose infinitesimal generators are given by Hermitian linear combinations of the operators:

$$\hat{C}^{(4)}_{\sigma\tau} = \sum_i^N a^\dagger_{\sigma i} a^{\tau i}, \qquad \hat{C}^{(N)}_{ij} = \sum_{\sigma=1}^4 a^\dagger_{\sigma i} a^{\sigma j}. \tag{4}$$

The group $U(N)$ is the group of one-body unitary transformations of the spatial wavefunctions and $U(4)$ is Wigner's supermultiplet group of one-body unitary transformations of the combined spin–isospin wavefunctions. An irrep of $U(N)$ carried by an A-nucleon subspace of $\mathbb{F}$ is labelled by an ordered partition $\{\lambda\} = \{\lambda_1\lambda_2\cdots\lambda_N\}$ of A, defined as a set of positive integers which add to A and satisfy the inequalities

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0. \tag{5}$$

An irrep of $U(4)$ carried by an A-nucleon subspace of $\mathbb{F}$ is similarly labelled by an ordered partition $\{\mu\} = \{\mu_1\mu_2\mu_3\mu_4\}$ of A, defined as a set of positive integers which likewise add to A and satisfy the inequalities

$$\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \mu_4 \geqslant 0. \tag{6}$$

As a result of the Schur–Weyl duality theorem, the total antisymmetry of the many-fermion wavefunctions implies that the states of $\mathbb{F}$ which belong to a $U(N)$ irrep $\{\lambda\}$ must simultaneously belong to the $U(4)$ irrep $\{\mu\} = \{\tilde{\lambda}\}$ conjugate to $\{\lambda\}$. The Young diagrams associated with conjugate partitions $\{\tilde{\lambda}\}$ and $\{\lambda\}$ transform into one another under reflection about a diagonal through the top-left corner; in other words, the number of boxes in the columns of the Young diagram for $\{\tilde{\lambda}\}$ are equal to the number of boxes in the corresponding rows of the diagram for $\{\lambda\}$. The unitary–unitary duality theorem further states that the Fock space $\mathbb{F}$ carries a reducible representation T of the group $U(4) \times U(N)$ given by the direct sum of irreps

$$T = \bigoplus_{A=0}^{4N} \bigoplus_{\lambda\vdash A} \{\tilde{\lambda}\} \times \{\lambda\}, \tag{7}$$

where $\lambda \vdash A$ signifies that $\{\lambda\}$ is an ordered partition of A. The partitions appearing in equation (7) are restricted by the condition that $\{\lambda\}$ and $\{\tilde{\lambda}\}$ have at most N and 4 parts, respectively; this means that $\tilde{\lambda}_1 \leqslant N$ and $\lambda_1 \leqslant 4$. It follows that $\mathbb{F}$ is a direct sum of A-nucleon Hilbert spaces, $\mathbb{F} = \bigoplus_{A=0}^{4N} \mathbb{H}^A$, with

$$\mathbb{H}^A = \bigoplus_{\lambda\vdash A} \mathbb{H}^{\{\lambda\}}, \tag{8}$$

where $\mathbb{H}^{\{\lambda\}}$ is the carrier space for the $U(4) \times U(N)$ irrep $\{\tilde{\lambda}\} \times \{\lambda\}$. The Hilbert space $\mathbb{H}^{\{\lambda\}}$ is uniquely characterized by the state

$$|\lambda\rangle = \left(a^\dagger_{11} a^\dagger_{21} \cdots a^\dagger_{\lambda_1 1}\right)\left(a^\dagger_{12} a^\dagger_{22} \cdots a^\dagger_{\lambda_2 2}\right) \cdots \left(a^\dagger_{1p} a^\dagger_{2p} \cdots a^\dagger_{\lambda_p p}\right)|0\rangle, \tag{9}$$

which is simultaneously of highest weight relative to both $U(4)$ and $U(N)$; $p = \tilde{\lambda}_1$ is the so-called *depth* of the partition $\{\lambda\}$, i.e., the number of parts of $\{\lambda\}$.

2.2. *Orthogonal–orthogonal duality*

The group of complex-linear transformations of the fermion operators of the form

$$a^\dagger_{\sigma i} \to \sum_{\tau j} \left(a^\dagger_{\tau j} u_{\tau j,\sigma i} + a^{\tau j} v_{\tau j,\sigma i}\right), \tag{10}$$

$$a^{\sigma i} \to \sum_{\tau j} \left(a^\dagger_{\tau j} v^*_{\tau j,\sigma i} + a^{\tau j} u^*_{\tau j,\sigma i}\right), \tag{11}$$

that preserve the fermion commutation relations of equation (3) is the orthogonal group $O(8N)$. This will be shown explicitly in [2]. The Lie algebra of this group is the so-called *fermion-pair algebra*. Its complex extension is spanned by the operators

$$a^\dagger_{\sigma i} a^\dagger_{\tau j}, \qquad a^{\sigma i} a^{\tau j}, \qquad \tfrac{1}{2}\left(a^\dagger_{\sigma i} a^{\tau j} - a^{\tau j} a^\dagger_{\sigma i}\right) = a^\dagger_{\sigma i} a^{\tau j} - \tfrac{1}{2}\delta^\sigma_\tau \delta^i_j. \tag{12}$$

We show in the following that this $O(8N)$ group has a pair of $O(8)$ and $O(N)$ subgroups with dual representations on the Fock space $\mathbb{F}$.

The subgroup $O(N)$ is defined as the set of all $U(N) \subset O(8N)$ transformations that leave the scalar products

$$a^\dagger_\sigma \cdot a^\dagger_\tau = \sum_i a^\dagger_{\sigma i} a^\dagger_{\tau \bar{\imath}} = -a^\dagger_\tau \cdot a^\dagger_\sigma, \qquad \sigma, \tau = 1, \ldots, 4 \tag{13}$$

invariant, where $\bar{\imath}$ (defined below) is an involution of the indices in which $i \to \bar{\imath}$ and $\bar{\imath} \to i$. The subgroup $O(8)$ is defined as the commutant of $O(N)$ in $O(8N)$, i.e., the set of all $O(8N)$ transformations that commute with those of $O(N)$.

A natural choice of the involution $i \to \bar{\imath}$ is given by time reversal. Let i index the orbital angular momentum quantum numbers $(l_i m_i)$ of single-particle states. For example, if $l_1 = 0$ and $l_2 = 1$ then a possible enumeration of the states is $i = 1 \equiv (0, 0)$, $i = 2 \equiv (1, -1)$, $i = 3 \equiv (1, 0)$ and $i = 4 \equiv (1, 1)$. A standard definition of time reversal is then given by setting

$$a^\dagger_{\sigma \bar{\imath}} = (-1)^{l_i + m_i} a^\dagger_{\sigma l_i, -m_i}. \tag{14}$$

With $a^\dagger_\sigma = \{a^\dagger_{\sigma l_i m_i}; m_i = -l_i, \ldots, l_i\}$ and $a^\dagger_\tau$ regarded as N-component vectors, the scalar product

$$\hat{\mathcal{A}}_{\sigma\tau} \equiv a^\dagger_\sigma \cdot a^\dagger_\tau = \sum_{l_i} \sum_{m_i = -l_i}^{l_i} \sqrt{2l_i + 1}(l_i, -m_i, l_i m_i|00) a^\dagger_{\sigma l_i m_i} a^\dagger_{\tau l_i, -m_i} \tag{15}$$

is then the creation operator of an angular-momentum-zero ($L = 0$) coupled pair.

For a single value l, $N = 2l+1$ is odd. One of the indices i is then associated with $m_i = 0$, the others occur in time-reverse pairs $(i, \bar{\imath})$ with $m_{\bar{\imath}} = -m_i$ for $m_i \neq 0$, and $\hat{\mathcal{A}}_{\sigma\tau}$ becomes $\sum_i a^\dagger_{\sigma i} a^\dagger_{\tau \bar{\imath}}$. However, when there is a multiplicity of l values, $N = \sum_{i=1}^k (2l_i + 1)$ may be even or odd and there is more than one single-particle state with $m_i = 0$. We next show that, for N even, there exists always a basis of single-nucleon states all occurring in time-reverse pairs and, for N odd, there exists a basis with just one unpaired single-nucleon state.

Suppose that $m_i = m_j = 0$, for some particular values of i and j, and that

$$a^\dagger_{\sigma\bar{\imath}} = a^\dagger_{\sigma i}, \qquad a^\dagger_{\sigma\bar{\jmath}} = a^\dagger_{\sigma j}. \tag{16}$$

The creation operators $a^\dagger_{\sigma i}$ and $a^\dagger_{\sigma j}$ can then be replaced by the linear combinations

$$\alpha^\dagger_{\sigma i} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma i} + \mathrm{i}a^\dagger_{\sigma j}\big), \qquad \alpha^\dagger_{\sigma j} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma i} - \mathrm{i}a^\dagger_{\sigma j}\big). \tag{17}$$

With the inclusion of complex conjugation of scalar operators in the definition of time reversal, the time reverse of the new operators is then given by

$$\alpha^\dagger_{\sigma\bar{\imath}} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma\bar{\imath}} - \mathrm{i}a^\dagger_{\sigma\bar{\jmath}}\big) = \alpha^\dagger_{\sigma j}, \qquad \alpha^\dagger_{\sigma\bar{\jmath}} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma\bar{\imath}} + \mathrm{i}a^\dagger_{\sigma\bar{\jmath}}\big) = \alpha^\dagger_{\sigma i}. \tag{18}$$

The new operators satisfy the desired anti-commutation relations

$$\big\{\alpha^{\sigma i}, \alpha^\dagger_{\tau j}\big\} = \delta^\sigma_\tau \delta^i_j, \qquad \big\{\alpha^\dagger_{\sigma i}, \alpha^\dagger_{\tau j}\big\} = \{\alpha^{\sigma i}, \alpha^{\tau j}\} = 0. \tag{19}$$

Moreover,

$$\alpha^\dagger_{\sigma i}\alpha^\dagger_{\tau\bar{\imath}} + \alpha^\dagger_{\sigma j}\alpha^\dagger_{\tau\bar{\jmath}} = a^\dagger_{\sigma i}a^\dagger_{\tau i} + a^\dagger_{\sigma j}a^\dagger_{\tau j} \tag{20}$$

as required. Thus, the unpaired creation operators $a^\dagger_{\sigma i}$ and $a^\dagger_{\sigma j}$ have been transformed into a time-reverse pair $\alpha^\dagger_{\sigma i}$ and $\alpha^\dagger_{\sigma\bar{\imath}} = \alpha^\dagger_{\sigma j}$. Similarly, if the initial $m_i = m_j = 0$ creation operators satisfy

$$a^\dagger_{\sigma\bar{\imath}} = a^\dagger_{\sigma i}, \qquad a^\dagger_{\sigma\bar{\jmath}} = -a^\dagger_{\sigma j}, \tag{21}$$

then the combinations

$$\alpha^\dagger_{\sigma i} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma i} + a^\dagger_{\sigma j}\big), \qquad \alpha^\dagger_{\sigma j} = \tfrac{1}{\sqrt{2}}\big(a^\dagger_{\sigma i} - a^\dagger_{\sigma j}\big) \tag{22}$$

give a time-reverse pair. Thus, without loss of generality, it may be assumed that, whenever N is even, there are no unpaired operators while, whenever N is odd, there is just one unpaired operator. In the following, we choose to index states such that $\bar{\imath} = N + 1 - i$ for all $i < (N + 1)/2$ and that, when N is odd, the index $i = \bar{\imath} = (N + 1)/2$ is the unpaired index.

As identified above, a set of infinitesimal generators of the group $U(N)$ is given by the operators $\big\{\hat{C}^{(N)}_{ij}\big\}$ of equation (4). Thus, the complex extension of the $O(N) \subset U(N)$ Lie algebra is spanned by linear combinations of these operators which commute with the $\{\hat{\mathcal{A}}_{\sigma\tau}\}$ scalars, e.g., the set

$$\hat{A}_{ij} = \hat{C}^{(N)}_{i\bar{\jmath}} - \hat{C}^{(N)}_{j\bar{\imath}}, \tag{23}$$

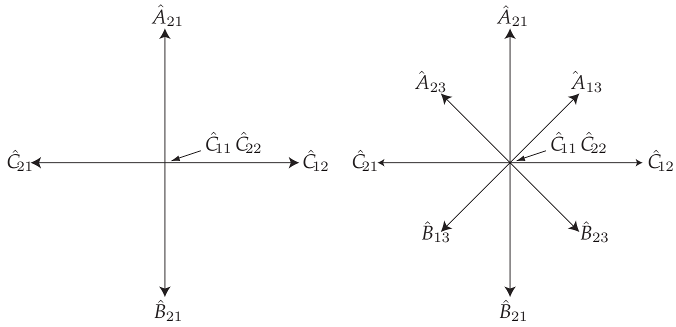
$$\hat{C}_{ij} = \hat{C}^{(N)}_{ij} - \hat{C}^{(N)}_{\bar{\jmath}\bar{\imath}}, \tag{24}$$

$$\hat{B}_{ij} = \hat{C}^{(N)}_{\bar{\jmath}i} - \hat{C}^{(N)}_{\bar{\imath}j}, \tag{25}$$

with i and j restricted to the interval $[1, \ldots, (N + 1)/2]$ if N is odd and to $[1, \ldots, N/2]$ if N is even. These operators are shown for $N = 4$ and 5 on the $SO(4)$ and $SO(5)$ root diagrams in figure 1. On the other hand, the Lie algebra of $O(8)$, the commutant of $O(N)$ in $O(8N)$, comprises all $O(N)$ scalars in the fermion-pair Lie algebra and is spanned by the operators

$$\hat{\mathcal{A}}_{\sigma\tau} = \sum_i a^\dagger_{\sigma i}a^\dagger_{\tau\bar{\imath}}, \tag{26}$$

$$\hat{\mathcal{B}}_{\sigma\tau} = \sum_i a^{\tau\bar{\imath}}a^{\sigma i}, \tag{27}$$



(*a*) so(4) (*b*) so(5)

Figure 1. The infinitesimal generators of equations (23)–(25), for $N = 4$ and 5, as root vectors for $SO(4)$ and $SO(5)$, respectively.

$$\hat{\mathcal{C}}_{\sigma\tau} = \sum_i a^\dagger_{\sigma i} a^{\tau i} - \frac{N}{2}\delta_{\sigma\tau} = \hat{C}^{(4)}_{\sigma\tau} - \frac{N}{2}\delta_{\sigma\tau}. \tag{28}$$

(Note that the Lie algebras of $O(N)$ and $O(8)$ are those of their $SO(N)$ and $SO(8)$ subgroups.)

In proving the duality between $O(8)$ and $O(N)$, it is helpful to visualize the situation by means of a schematic diagram. Each row of lines in figure 2 represents a Hilbert space $\mathbb{H}^{\{\lambda^{(k)}\}}$ for an A_k-nucleon $U(4) \times U(N)$ irrep $\{\lambda^{(k)}\}$ with $\{\lambda^{(k)}\} \vdash A_k$ and A_k increasing in steps of two in ascending order, i.e., $A_{k+1} = A_k + 2$. The lines in a given row represent the irreducible $U(4) \times O(N)$ subspaces $\mathbb{H}^{\{\tilde{\lambda}^{(k)}\}}_{[\kappa^{(i)}]} \subset \mathbb{H}^{\{\lambda^{(k)}\}}$, in the decomposition

$$\mathbb{H}^{\{\lambda^{(k)}\}} = \bigoplus_i c^{\lambda^{(k)}}_{\kappa^{(i)}} \mathbb{H}^{\{\tilde{\lambda}^{(k)}\}}_{[\kappa^{(i)}]} \tag{29}$$

where $\bigoplus$ denotes a direct sum of subspaces for $U(4) \times O(N)$ irreps $\{\tilde{\lambda}^{(k)}\} \times [\kappa^{(i)}]$ with multiplicity $c^{\lambda^{(k)}}_{\kappa^{(i)}}$ and the sum runs over the $O(N)$ irrep labels $[\kappa]$ given by the branching rule

$$U(N) \downarrow O(N) : \{\lambda\} \downarrow \sum_\kappa c^\lambda_\kappa [\kappa]. \tag{30}$$

Note that an irrep $[\kappa]$ may appear in the expansion of different $\{\lambda\}$. Lines representing $U(4) \times O(N)$ subspaces with a common $O(N)$ label $[\kappa]$ are arranged, in the diagram, into columns.

We now claim that all the states in the Fock space $\mathbb{F} = \bigoplus \mathbb{H}^A$ which belong to equivalent $[\kappa]$ irreps of $O(N)$ span an $O(8) \times O(N)$ irrep, i.e., that all the lines in a column (cf figure 2) represent $U(4) \times O(N)$ subspaces of an $O(8) \times O(N)$ irrep. The content of this claim is clarified in figure 3 for $N = 2$.

In validating this claim, it is convenient to label an $O(8)$ irrep by $\left[\frac{1}{2}N(\nu)\right]$. This (non-standard) notation defines an $O(8)$ irrep in terms of a lowest weight state which satisfies the equations

$$\hat{\mathcal{C}}_{\sigma\tau}\left|\left[\tfrac{1}{2}N(\nu)\right]; \text{l.wt.}\right\rangle = 0, \qquad \text{for} \quad 1 \leqslant \sigma < \tau \leqslant 4, \tag{31}$$

$$\hat{\mathcal{B}}_{\sigma\tau}\left|\left[\tfrac{1}{2}N(\nu)\right]; \text{l.wt.}\right\rangle = 0, \qquad \text{for} \quad 1 \leqslant \sigma \leqslant \tau \leqslant 4, \tag{32}$$

$$\hat{\mathcal{C}}_{\sigma\sigma}\left|\left[\tfrac{1}{2}N(\nu)\right]; \text{l.wt.}\right\rangle = \left(\nu_\sigma - \tfrac{1}{2}N\right)\left|\left[\tfrac{1}{2}N(\nu)\right]; \text{l.wt.}\right\rangle, \qquad \text{for} \quad 1 \leqslant \sigma \leqslant 4, \tag{33}$$

where $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ is a $U(4)$ weight defined by the operators $\hat{C}^{(4)}_{\sigma\sigma}$, cf equation (4). With this notation, we proceed to establish the $O(8) \leftrightarrow O(N)$ duality relationship by showing that

	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mathbb{H}^{\{\lambda^{(5)}\}}$	$\overline{\{\tilde\lambda^{(5)}\}\times[\kappa^{(1)}]}$	$\overline{\{\tilde\lambda^{(5)}\}\times[\kappa^{(2)}]}$	$\overline{\{\tilde\lambda^{(5)}\}\times[\kappa^{(3)}]}$	$\overline{\{\tilde\lambda^{(5)}\}\times[\kappa^{(4)}]}$	$\overline{\{\tilde\lambda^{(5)}\}\times[\kappa^{(5)}]}$
$\mathbb{H}^{\{\lambda^{(4)}\}}$	$\overline{\{\tilde\lambda^{(4)}\}\times[\kappa^{(1)}]}$	$\overline{\{\tilde\lambda^{(4)}\}\times[\kappa^{(2)}]}$	$\overline{\{\tilde\lambda^{(4)}\}\times[\kappa^{(3)}]}$	$\overline{\{\tilde\lambda^{(4)}\}\times[\kappa^{(4)}]}$	
$\mathbb{H}^{\{\lambda^{(3)}\}}$	$\overline{\{\tilde\lambda^{(3)}\}\times[\kappa^{(1)}]}$	$\overline{\{\tilde\lambda^{(3)}\}\times[\kappa^{(2)}]}$	$\overline{\{\tilde\lambda^{(3)}\}\times[\kappa^{(3)}]}$	$\overline{\{\tilde\lambda^{(3)}\}\times[\kappa^{(4)}]}$	
$\mathbb{H}^{\{\lambda^{(2)}\}}$	$\overline{\{\tilde\lambda^{(2)}\}\times[\kappa^{(1)}]}$	$\overline{\{\tilde\lambda^{(2)}\}\times[\kappa^{(2)}]}$			
$\mathbb{H}^{\{\lambda^{(1)}\}}$	$\overline{\{\tilde\lambda^{(1)}\}\times[\kappa^{(1)}]}$				

Figure 2. A schematic representation of the decomposition of a selected set of $\mathbb{H}^{\{\lambda^{(k)}\}}$ spaces into $U(4)\times O(N)$ subspaces. Each line represents a multiplicity $c^{\lambda^{(k)}}_{\kappa^{(i)}} \geqslant 1$, of $U(4)\times O(N)$ irreps equivalent to $\{\tilde\lambda^{(k)}\}\times[\kappa^{(i)}]$ in the corresponding $\mathbb{H}^{\{\lambda^{(k)}\}}$.

	$\{\lambda\}$				$\{\lambda\}$			$\{\lambda\}$	
$A=8$	$\{4^2\}$	$\overline{\{2^4\}\times[0]}$							
$A=6$	$\{42\}$	$\overline{\{2^21^2\}\times[0]}$	$\overline{\{2^21^2\}\times[2]}$		$\{3^2\}$	$\overline{\{2^3\}\times[2]}$			
$A=4$	$\{4\}$	$\overline{\{1^4\}\times[0]}$	$\overline{\{1^4\}\times[2]}$	$\overline{\{1^4\}\times[4]}$	$\{31\}$	$\overline{\{21^2\}\times[2]}$	$\overline{\{21^2\}\times[1^2]}$	$\{2^2\}$	$\overline{\{2^2\}\times[0]}$
$A=2$	$\{2\}$	$\overline{\{1^2\}\times[0]}$	$\overline{\{1^2\}\times[2]}$		$\{1^2\}$	$\overline{\{2\}\times[2]}$			
$A=0$	$\{0\}$	$\overline{\{0\}\times[0]}$							
	$(\tilde\kappa)$	$[0]$	$[1^2]$	$[1^4]$		$[1^2]$	$[2]$		$[0]$

Figure 3. The spectrum of $U(4)\times O(N)$ irreps, $\{\tilde\lambda\}\times[\kappa]$, for $N=2$. The set of all $U(4)\times O(N)$ irreps with common $[\kappa]$ is contained in a single $O(8)\times O(N)$ irrep $[1(\tilde\kappa)]\times[\kappa]$.

every $O(8)$ irrep $\left[\frac{1}{2}N(\nu)\right]$ is associated with a column of $U(4)\times O(N)$ irreps and is uniquely defined by the $O(N)$ label $[\kappa]$ for the column. It then follows that $(\nu)=(\tilde\kappa)$ and that there is a dual pair of subgroup chains with irrep labels given by

$$\begin{matrix} O(8) & \supset & U(4), & \quad & U(N) & \supset & O(N) \\ \left[\frac{1}{2}N(\tilde\kappa)\right] & & \{\tilde\lambda\} & & \{\lambda\} & & [\kappa] \end{matrix}. \tag{34}$$

Our strategy is to consider first the highest weight states $\{|\tilde\lambda\kappa\rangle\}$ for the $U(4)\times O(N)$ subspaces and subsequently identify the subset of these states that are also of $O(8)$ lowest weight. To be a $U_{(4)}$ highest weight state $|\tilde\lambda\kappa\rangle$ is required to satisfy the equations

$$\hat C^{(4)}_{\sigma\tau}|\tilde\lambda\kappa\rangle = \begin{cases} 0 & \text{for} \quad 1\leqslant\sigma<\tau\leqslant 4, \\ \tilde\lambda_\sigma|\tilde\lambda\kappa\rangle & \text{for} \quad 1\leqslant\sigma=\tau\leqslant 4. \end{cases} \tag{35}$$

Note that the special state $|\lambda\rangle$, defined by equation (9) to be a $U(N)$ highest weight state, also satisfies the equations

$$\hat C_{ij}|\lambda\rangle = 0, \qquad 0<i<j, \tag{36}$$

$$\hat{A}_{ij}|\lambda\rangle = 0, \qquad \forall i, j \leqslant N/2, \tag{37}$$

which means that it is also a highest weight state for $O(N)$ and $SO(N)$. The components of the $SO(N)$ weight, $[\kappa^{(\lambda)}]$, for this state $|\lambda\rangle$ are given by the eigenvalues of the Cartan operators $\{\hat{C}_{ii}\}$:

$$\kappa_i^{(\lambda)}|\lambda\rangle = \hat{C}_{ii}|\lambda\rangle = \left(\hat{C}_{ii}^{(N)} - \hat{C}_{\bar{i}\bar{i}}^{(N)}\right)|\lambda\rangle = (\lambda_i - \lambda_{\bar{i}})|\lambda\rangle, \qquad i \leqslant N/2, \tag{38}$$

i.e., $\kappa_i^{(\lambda)} = \lambda_i - \lambda_{\bar{i}}$. The corresponding $O(N)$ irrep $[\kappa^{(\lambda)}]$ with highest weight state $|\lambda\rangle$ is defined below. The important point, for the moment, is to note that the state $|\lambda\rangle \equiv |\tilde{\lambda}\kappa^{(\lambda)}\rangle$ defines an irreducible $U(4) \times O(N)$ subspace $\mathbb{H}^{\{\tilde{\lambda}\}}_{[\kappa^{(\lambda)}]} \subset \mathbb{H}^{\{\lambda\}}$, which, in the diagram of figure 2, is associated with the line at the far right of the row for $\mathbb{H}^{\{\lambda\}}$.

We now claim that only a state, $|\lambda\rangle$, in the set of $U(4) \times U(N)$ highest weight states, can be both an $O(N)$ highest and an $O(8)$ lowest weight state. This claim is based on the observation that any state $|\tilde{\lambda}\kappa\rangle$ of $U(4)$ and $O(N)$ highest weight in the space $\mathbb{H}^{\{\lambda\}}$, that is not also a $U(N)$ highest weight state, is of the form

$$|\tilde{\lambda}\kappa\rangle = [\hat{\mathcal{A}} \otimes |\tilde{\lambda}'\kappa\rangle]^{\{\tilde{\lambda}\}}_{U(4)h.wt.}, \tag{39}$$

for some λ', where the bracket signifies a $U(4)$ coupling $\{1^2\} \otimes \{\tilde{\lambda}'\} \to \{\tilde{\lambda}\}$. This follows because the $\{\hat{\mathcal{A}}_{\sigma\tau}\}$ operators are $O(8)$ raising operators; hence, none of the states $\{|\tilde{\lambda}\kappa\rangle\}$ with $\kappa \neq \kappa^{(\lambda)}$ can be $O(8)$ lowest weight states. Thus, we need only look among the set $\{|\lambda\rangle \equiv |\tilde{\lambda}\kappa^{(\lambda)}\rangle\}$, to find states in $\mathbb{H}^A$ that are simultaneously of lowest weight in $O(8)$ and highest weight in $O(N)$.

To be an $O(8)$ lowest weight state, $|\lambda\rangle$ has to satisfy the condition

$$\hat{\mathcal{B}}_{\sigma\tau}|\lambda\rangle = 0, \qquad \forall \sigma < \tau. \tag{40}$$

First observe that

$$\left[\hat{\mathcal{B}}_{\sigma\tau}, a^\dagger_{\rho i}\right] = \delta_{\sigma\rho} a^{\tau\bar{i}} - \delta_{\tau\rho} a^{\sigma\bar{i}}. \tag{41}$$

With the expression for $|\lambda\rangle$, given by equation (9), reexpressed (to within a $\pm$ sign) in the form

$$|\lambda\rangle = \left[\left(a^\dagger_{11}\cdots a^\dagger_{1\tilde{\lambda}_1}\right)\left(a^\dagger_{21}\cdots a^\dagger_{2\tilde{\lambda}_2}\right)\cdots\right]|0\rangle, \tag{42}$$

it follows that

$$\begin{aligned}
\hat{\mathcal{B}}_{\sigma\tau}|\lambda\rangle &= \sum_i \left[\left(a^\dagger_{11}\cdots a^\dagger_{1\tilde{\lambda}_1}\right)\cdots\left(a^\dagger_{\sigma 1}\cdots a^\dagger_{\sigma i-1} a^{\tau\bar{i}} a^\dagger_{\sigma i+1}\cdots a^\dagger_{\sigma\tilde{\lambda}_\sigma}\right)\cdots\left(a^\dagger_{\tau 1}\cdots a^\dagger_{\tau\bar{i}}\cdots a^\dagger_{\tau\tilde{\lambda}_\tau}\right)\cdots\right]|0\rangle \\
&\quad - \sum_i \left[\left(a^\dagger_{11}\cdots a^\dagger_{1\tilde{\lambda}_1}\right)\cdots\left(a^\dagger_{\sigma 1}\cdots a^\dagger_{\sigma\bar{i}}\cdots a^\dagger_{\sigma\tilde{\lambda}_\sigma}\right)\cdots\left(a^\dagger_{\tau 1}\cdots a^\dagger_{\tau i-1} a^{\sigma\bar{i}} a^\dagger_{\tau i+1}\cdots a^\dagger_{\tau\tilde{\lambda}_\tau}\right)\cdots\right]|0\rangle \\
&= \pm\sum_i \left[\left(a^\dagger_{11}\cdots a^\dagger_{1\tilde{b}_1}\right)\cdots\left(a^\dagger_{\sigma 1}\cdots a^\dagger_{\sigma i-1} a^\dagger_{\sigma i+1}\cdots a^\dagger_{\sigma\tilde{\lambda}_\sigma}\right)\cdots\right. \\
&\quad \times \left.\left(a^\dagger_{\tau 1}\cdots a^\dagger_{\tau\bar{i}-1} a^\dagger_{\tau\bar{i}+1}\cdots a^\dagger_{\tau\tilde{b}_\tau}\right)\cdots\right] a^{\tau\bar{i}} a^\dagger_{\tau\bar{i}}|0\rangle \\
&\mp \sum_i \left[\left(a^\dagger_{11}\cdots a^\dagger_{1\tilde{\kappa}_1}\right)\cdots\left(a^\dagger_{\sigma 1}\cdots a^\dagger_{\sigma\bar{i}-1} a^\dagger_{\sigma\bar{i}+1}\cdots a^\dagger_{\sigma\tilde{\kappa}_\sigma}\right)\cdots\right. \\
&\quad \times \left.\left(a^\dagger_{\tau 1}\cdots a^\dagger_{\tau i-1} a^\dagger_{\tau i+1}\cdots a^\dagger_{\tau\tilde{\kappa}_\tau}\right)\cdots\right] a^\dagger_{\sigma\bar{i}} a^{\sigma\bar{i}}|0\rangle.
\end{aligned} \tag{43}$$

Each term in the second sum of equation (43) clearly vanishes. The remaining terms are all linearly independent and so $\hat{\mathcal{B}}_{\sigma\tau}|\lambda\rangle$ will be non-zero *if and only if* there is a value of i for which $i \leqslant \tilde{\lambda}_\sigma$ and $\bar{i} = N + 1 - i \leqslant \tilde{\lambda}_\tau$, i.e., if and only if $\tilde{\lambda}_\sigma + \tilde{\lambda}_\tau \geqslant N + 1$, for some σ and

τ. The largest value of $\tilde{\lambda}_\sigma + \tilde{\lambda}_\tau$ is $\tilde{\lambda}_1 + \tilde{\lambda}_2$. Thus, the condition that $\hat{\mathcal{B}}_{\sigma\tau}|\lambda\rangle = 0$ and $|\lambda\rangle$ is an $O(8)$ lowest weight state is that

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N. \tag{44}$$

Thus, we have shown that every state in the Fock space $\mathbb{F}$ that is simultaneously a lowest weight state of an $O(8)$ irrep and a highest weight state of an $O(N)$ irrep is also a $U(4)\times U(N)$ highest weight state $|\lambda\rangle$ with $\tilde{\lambda}_1+\tilde{\lambda}_2 \leqslant N$. Conversely, we have shown that every $U(4)\times U(N)$ highest weight state $|\lambda\rangle$ with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N$ is simultaneously a lowest weight state of an $O(8)$ irrep and a highest weight state of an $O(N)$ irrep. The unitary–unitary duality relationship (7) implies that each state $|\lambda\rangle$ is uniquely defined, without multiplicity, by the partition $\{\lambda\}$. To prove the duality relationship between the $O(8)$ and $O(N)$ irreps, it remains to show that different $\{\lambda\}$, with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N$, define different $O(8)$ and different $O(N)$ irreps (cf condition (iii) of the definition of dual representations).

Note that within the subset of $U(N)$ highest weights $\{\lambda\}$ with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N$ there are pairs of distinct $U(N)$ highest weights (λ, λ') for which $\kappa^{(\lambda)} = \kappa^{(\lambda')}$, e.g., the $U(2)$ irreps $\{0\}$ and $\{1^2\}$. Recall that the $U(N)$ highest weight state $|\lambda\rangle$ has $SO(N)$ weight $[\kappa^{(\lambda)}]$ with $\kappa_i^{(\lambda)} = \lambda_i - \lambda_{\bar{i}}$. However, it is stressed that the labels for an $SO(N)$ irrep alone are not generally sufficient to distinguish different $O(N)$ irreps. Thus, in a notation introduced by Murnaghan [23], an asterisk is used to distinguish $O(N)$ irreps which correspond to the same $SO(N)$ irrep $[\kappa]$ but differ by a factor $\det(g) = \pm 1$ in the representation of an element $g \in O(N)$. (Note that, because $SO(N)$ is the subgroup of elements in $O(N)$ for which $\det(g) = 1$, the two irreps $[\kappa]$ and $[\kappa]^*$ become equivalent on restriction to $SO(N)$.)

We now explain the meaning of the asterisk in the present context and show that if one member of a (λ, λ') pair, for which $\kappa^{(\lambda)} = \kappa^{(\lambda')}$ but $\lambda \neq \lambda'$, defines an $O(N)$ irrep $[\kappa]$, then the other member defines the irrep $[\kappa]^*$. We start from the observation that $[\kappa^{(\lambda)}] = [\lambda]$ whenever $\{\lambda\}$ is a $U(N)$ irrep for which $\tilde{\lambda}_1 \leqslant N/2$ and that $[\kappa^{(\lambda)}] = [\kappa^{(\lambda')}]$ when $\{\lambda'\}$ is the related $U(N)$ irrep defined such that $\tilde{\lambda}'_1 = N - \tilde{\lambda}_1$ and $\tilde{\lambda}'_i = \tilde{\lambda}_i$ for $i > 1$.

First observe that the group $O(N)$ has two one-dimensional irreps: one spanned, for example, by the fermion vacuum state $|0\rangle$ and one spanned by the state

$$|1^N\rangle = \big(a^\dagger_{11} \cdots a^\dagger_{1N}\big)|0\rangle. \tag{45}$$

The former is the identity irrep and the latter is the irrep in which an element $g \in O(N)$ is represented by its determinant $\det(g)$. These two $O(N)$ irreps are labelled [0] and [0]*, respectively. Next observe that a state

$$|1^n\rangle = \big(a^\dagger_{11} \cdots a^\dagger_{1n}\big)|0\rangle \tag{46}$$

with $n < N$ can be reexpressed, to within a $\pm$ sign, as

$$|1^n\rangle = \pm(a^{1\bar{1}} \cdots a^{1\bar{m}})|1^N\rangle, \tag{47}$$

where $m = N - n$. Moreover, because the operators $\sum_i a^\dagger_{1i} a^\dagger_{1\bar{i}}$ and $\sum_i a^\dagger_{1\bar{i}} a^{1\bar{i}}$ are both $O(N)$ scalars, the operators $a^{1\bar{i}}$ and $a^\dagger_{1i}$ transform in the same way under $O(N)$. It follows that the operator $(a^{1\bar{1}} \cdots a^{1\bar{m}})$ is the highest weight component of a $[1^m]$ tensor operator. Thus, when $n > N/2$, the $O(N)$ irrep with highest weight state $|1^n\rangle$ is the irrep $[1^{N-n}]^*$. Similar considerations apply to $O(N)$ irreps with highest weight states $|\lambda\rangle$ and $|\lambda'\rangle$ for which $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N$ and $\tilde{\lambda}'_1 + \tilde{\lambda}'_2 \leqslant N$ but while $\tilde{\lambda}_1 \leqslant N/2$, $\tilde{\lambda}'_1 > N/2$. By expressing the state $|\lambda'\rangle$ in the form

$$|\lambda'\rangle = \pm(a^{1\bar{1}} \cdots a^{1\bar{m}})\big[\big(a^\dagger_{21} \cdots a^\dagger_{2\tilde{\lambda}'_2}\big) \cdots\big]|1^N\rangle, \tag{48}$$

with $m = N - \tilde{\lambda}'_1$, it follows that the $O(N)$ irrep with highest weight state $|\lambda'\rangle$ is the irrep $[\kappa^{(\lambda)}]^*$ with $\tilde{\lambda}'_1 = N - \tilde{\lambda}_1$ and $\tilde{\lambda}'_i = \tilde{\lambda}_i$ for $i > 1$. However, for present purposes, it will be convenient to follow Littlewood's convention [24], rather than that of Murnaghan, and label an $O(N)$ irrep, whose highest weight state is also a $U(N)$ highest weight state $|\lambda\rangle$ with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leqslant N$, simply by $[\lambda]$ (even if $\tilde{\lambda}_1 > N/2$) and avoid the use of asterisks.

Thus, we have substantiated the $O(8) \times O(N)$ duality relationship and, moreover, determined that if Γ is the reducible representation of $O(8) \times O(N)$ carried by the Fock space $\mathbb{F}$, then

$$\Gamma = \bigoplus_{A=0}^{4N} \bigoplus_{\kappa \vdash A} \left[\tfrac{1}{2} N(\tilde{\kappa})\right] \times [\kappa], \tag{49}$$

where $\kappa \vdash A$ is restricted by the conditions $\kappa_1 \leqslant 4$ and $\tilde{\kappa}_1 + \tilde{\kappa}_2 \leqslant N$ and it is understood that when $\tilde{\kappa}_1 > N/2$ then $[\kappa] \equiv [\mathrm{\kappa}]^*$ with $\tilde{\mathrm{\kappa}}_1 = N - \tilde{\kappa}_1$ and $\tilde{\mathrm{\kappa}}_i = \tilde{\kappa}_i$ for $i > 1$. (The parallel complication does not arise for the $O(8)$ irreps labelled here by $\left[\frac{1}{2} N(\tilde{\kappa})\right]$ because of the restriction $\kappa_1 \leqslant 4$.)

2.3. The duality relationship between the $U(N) \downarrow O(N)$ and $O(8) \downarrow U(4)$ branching rules

A dual pair of subgroup chains implies a useful relationship between the branching rules of groups in the paired chains. In the present situation, the reduction of the $U(4) \times U(N)$ representation $T = \bigoplus_{A=0}^{4N} \bigoplus_{\lambda \vdash A} \{\tilde{\lambda}\} \times \{\lambda\}$, given by equation (7), and the reduction of the $O(8) \times O(N)$ representation $\Gamma = \bigoplus_{A=0}^{4N} \bigoplus_{\kappa \vdash A} \left(\frac{1}{2} N[\tilde{\kappa}]\right) \times [\kappa]$, given by equation (49), imply that the $U(N) \downarrow O(N)$ and $O(8) \downarrow U(4)$ branching rules are related.

If the $U(N) \downarrow O(N)$ branching rule is expressed by a set of $R_{\lambda\kappa}$ coefficients

$$U(N) \downarrow O(N) : \{\lambda\} \downarrow \bigoplus_{\kappa} R_{\lambda\kappa} [\kappa], \tag{50}$$

then the $U(4) \times U(N)$ representation T restricts to the $U(4) \times O(N)$ representation

$$U(4) \times U(N) \downarrow U(4) \times O(N) : T \downarrow \bigoplus_{A=0}^{4N} \bigoplus_{\lambda \vdash A} \bigoplus_{\kappa} R_{\lambda\kappa} \{\tilde{\lambda}\} \times [\kappa]. \tag{51}$$

Similarly, if the $O(8) \downarrow U(4)$ branching rule is expressed by

$$O(8) \downarrow U(4) : \left(\tfrac{1}{2} N[\tilde{\kappa}]\right) \downarrow \bigoplus_{\lambda} \mathcal{R}_{\lambda\kappa} \{\tilde{\lambda}\}, \tag{52}$$

then the $O(8) \times O(N)$ representation Γ restricts to the $U(4) \times O(N)$ representation

$$O(8) \times O(N) \downarrow U(4) \times O(N) : \Gamma \downarrow \bigoplus_{A=0}^{4N} \bigoplus_{\lambda \vdash A} \bigoplus_{\kappa} \mathcal{R}_{\lambda\kappa} \{\tilde{\lambda}\} \times [\kappa]. \tag{53}$$

Because the representations T and Γ share the same Hilbert space, their restrictions to their identical $U(4) \times O(N)$ subgroup must be the same. Thus, it follows that the coefficients of the $U(N) \downarrow O(N)$ and $O(8) \downarrow U(4)$ branching rules are identical, i.e.,

$$\mathcal{R}_{\lambda\kappa} = R_{\lambda\kappa}. \tag{54}$$

The $U(N) \downarrow O(N)$ branching rules are well known and given by King [26]. Table 1 gives the $R_{\lambda\kappa}$ coefficients for the $U(3)$ irreps for which the duality relationship (54) applies, i.e., those for which $\lambda_1 \leqslant 4$. Note that, in general, two $U(N)$ irreps $\{\lambda^{(1)}\}$ and $\{\lambda^{(2)}\}$ restrict to the same $O(N)$ representation whenever $\lambda_i^{(1)} + \lambda_{N+1-i}^{(2)} = k$ (with k even) for all $i = 1$ to N. Thus, for example (cf table 1), the $U(3)$ irreps $\{2\}$, $\{2^2\}$, $\{4^2 2\}$ and $\{4, 2^2\}$ all branch to the same sum of $O(3)$ irreps. Tables of $O(8) \downarrow U(4)$ branching rules for $N \leqslant 6$ are given in the appendix.

Table 1. The non-zero $R_{\lambda\kappa}$ coefficients of the $U(3) \downarrow O(3)$ branching rule.

$[\kappa]$	$\{0\}$ $\{4^3\}$	$\{2\}$ $\{4^22\}$	$\{1^2\}$ $\{43^2\}$	$\{4\}$ $\{4^2\}$	$\{31\}$ $\{431\}$	$\{2^2\}$ $\{42^2\}$	$\{21^2\}$ $\{3^22\}$	$\{42\}$	$\{41^2\}$ $\{3^2\}$	$\{321\}$	$\{2^3\}$
$[0]$	1	1		1		1		1			1
$[2]$		1		1	1	1		2		1	
$[1^2] \equiv [0]^*$			1		1		1		1	1	
$[4]$				1				1			
$[31] \equiv [3]^*$					1			1	1		

$[\kappa]$	$\{1\}$ $\{4^23\}$	$\{3\}$ $\{4^21\}$	$\{21\}$ $\{432\}$	$\{1^3\}$ $\{3^3\}$	$\{41\}$ $\{43\}$	$\{32\}$ $\{421\}$	$\{31^2\}$ $\{3^21\}$	$\{2^21\}$ $\{32^2\}$
$[1]$	1	1	1		1	1		1
$[3]$		1			1	1		
$[21] \equiv [2]^*$			1		1	1	1	
$[1^3]$				1			1	
$[41] \equiv [4]^*$					1			

3. An application to the nuclear 2s1d shell in LST-coupling

Nuclear shell-model calculations are most commonly carried out in a jj-coupled basis. However, for many purposes there are considerable advantages to an LST-coupled basis. An LST-coupled basis is any basis that reduces the subgroup chain

$$U(4) \times U(N) \supset SU(2)_T \times SU(2)_S \times SO(3)_L \supset SU(2)_J, \tag{55}$$

where $SO(3)_L$ is the orbital angular momentum group, $SU(2)_S$ and $SU(2)_T$ are respectively the spin and isospin subgroups of the Wigner supermultiplet group $U(4)$, and $SU(2)_J$ is the total orbital-angular-momentum-plus-spin subgroup of $SO(3)_L \times SU(2)_S$. (In regarding $SU(2)_J$ as a subgroup of $SO(3)_L \times SU(2)_S$ we use the standard device of extending $SO(3)_L$ to its $SU(2)_L$ covering group.) However, there are several possibilities within this skeletal framework. For example, for the 2s1d shell, for which $N = 6$, there are three more detailed coupling schemes given by classifying basis states by the quantum numbers of the subgroups chains:

$$\begin{array}{ccccccc} U(6) & \supset & U(3) & \supset & SU(3) & \supset & SO(3) \\ \{\Lambda\} & & \{\lambda_1\lambda_2\lambda_3\} & & (\lambda\mu) & & L \end{array}, \tag{56}$$

$$\begin{array}{ccccccc} U(6) & \supset & O(6) & \supset & O(5) & \supset & SO(3) \\ \{\Lambda\} & & [\kappa] & & [v] & & L \end{array}, \tag{57}$$

$$\begin{array}{ccccccc} U(6) & \supset & U(5) & \supset & O(5) & \supset & SO(3) \\ \{\Lambda\} & & \{\mu\} & & [v] & & L \end{array}, \tag{58}$$

where we now label irreps by the symbols given below each group.

It is notable that these subgroup chains are the same as those of the interacting boson model. However, because they are now used to classify the states of an interacting fermion system, their physical content is different. Moreover, the $U(6)$ irreps $\{\Lambda\}$ that appear in the classification of the nuclear 2s1d shell are those whose conjugates, $\{\tilde{\Lambda}\}$, define $U(4)$ irreps, i.e., $\{\Lambda\}$ is a partition of maximum length 4 and maximum depth 6.

As we now show, each subgroup chain diagonalizes a two-body Hamiltonian of physical interest.

3.1. The U(3) chain

The $U(3)$ chain of equation (56) gives the basis of $SU(3)$-coupled states used by Elliott [27] in his model of nuclear rotational states. This chain diagonalizes the so-called $\hat{Q}\cdot\hat{Q}$ interaction which is given, to within a term in the square of the $SO(3)$ angular momentum, by the (arbitrarily normalized) $SU(3)$ Casimir operator

$$\hat{C}_{SU3} = \tfrac{1}{4}\left[\hat{Q}\cdot\hat{Q} + 3\hat{L}\cdot\hat{L}\right]. \tag{59}$$

The value of the Casimir operator for an SU(3) irrep is given (with this normalization) by

$$\langle(\lambda\mu)|\hat{C}_{SU3}|(\lambda\mu)\rangle = \lambda^2 + \lambda\mu + \mu^2 + 3(\lambda+\mu). \tag{60}$$

On restriction to $U(3)$, the $U(6)$ irrep $\{1\}$ restricts to the $U(3)$ irrep $\{2\}$. More generally, the $U(6)\downarrow U(3)$ branching rule is given by the plethysm

$$U(6)\downarrow U(3) : \{\Lambda\}\downarrow\{2\} \circledP \{\Lambda\}. \tag{61}$$

The further restriction to $SU(3)$ is given by

$$U(3)\downarrow SU(3) : \{\lambda_1\lambda_2\lambda_3\}\downarrow(\lambda_1-\lambda_2, \lambda_2-\lambda_3). \tag{62}$$

The $SU(3)$ irreps contained in the $U(6)$ irreps that occur in the 2s1d shell have been given by Elliott [27].

The spectra of the Hamiltonian

$$\hat{H} = -\hat{C}_{SU3} \tag{63}$$

for 2s1d-shell nuclei with supermultiplet symmetry $\{n^4\}$ for $n = 1, \ldots, 6$ are shown in figure 4.

The $SU(3)\downarrow SO(3)$ branching rule [27] is given by

$$\begin{aligned} SU(3)\downarrow SO(3) : (\lambda,\mu)\downarrow L &= K, K+1, \ldots, K+\lambda, && \text{for} \quad K\neq 0,\\ &= \lambda, \lambda-2, \ldots, 1 \text{ or } 0, && \text{for} \quad K = 0,\\ K &= \mu, \mu-2, \ldots, 1 \text{ or } 0, \end{aligned} \tag{64}$$

Thus, it is straightforward to include the L-dependence of the energy-level spectra for a Hamiltonian $\hat{H} = -\chi\hat{Q}\cdot\hat{Q}$ or, more generally, to give the spectrum of many rotor-like bands for a Hamiltonian of the form

$$\hat{H} = \chi_1\hat{C}_{SU3} + \chi_2\hat{L}\cdot\hat{L}. \tag{65}$$

3.2. The O(6) chain

In light of the duality between the $U(N)\downarrow O(N)$ and $O(8)\supset U(4)$ subgroup chains, the $O(6)$ chain of equation (57) gives rise to a coupling scheme, which diagonalizes an $L = 0$ pairing interaction. In contrast to the $J = 0$ pairing interaction in a multi-j-shell configuration, which gives rise to generalized seniority, the $L = 0$ pairing interaction acts in multi-l-shell configurations and preserves supermultiplet symmetry. It is expressed in terms of the dual $O(8)$ raising and lowering operators of equation (26) and (27) by

$$\hat{V}^P_{O6} = \sum_{\sigma,\tau}\hat{A}_{\sigma\tau}\hat{B}_{\sigma\tau} \tag{66}$$

and is expressed in terms of the $U(4)$ and $O(8)$ Casimir operators as follows.

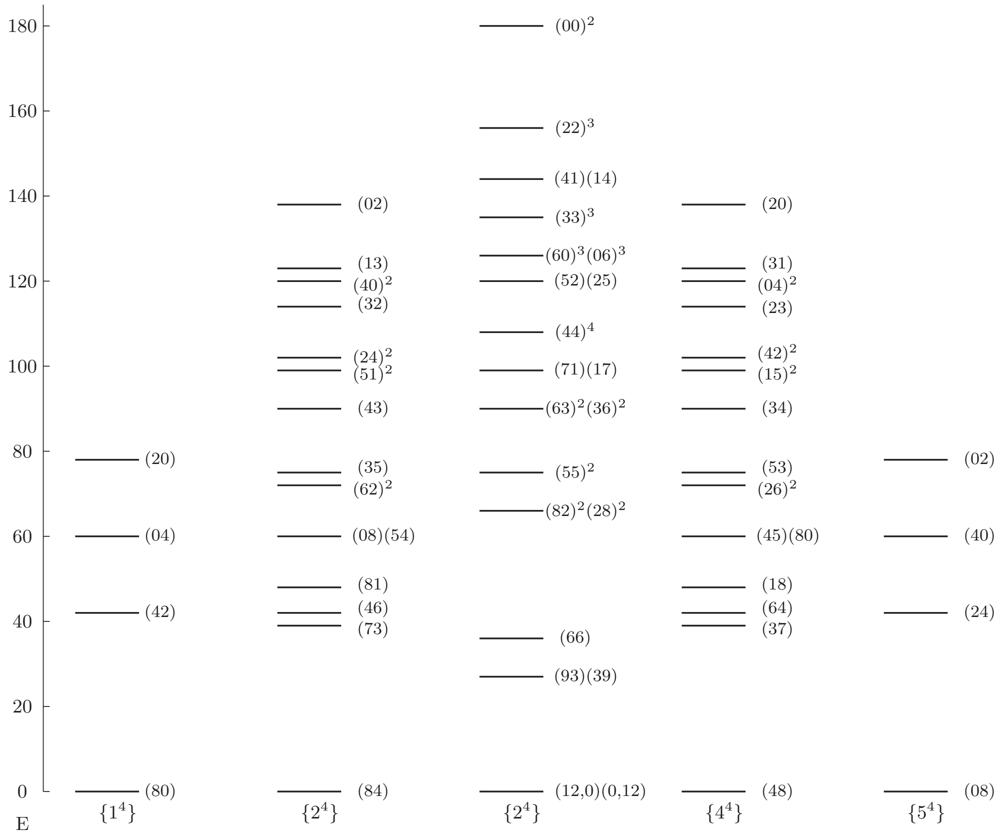


Figure 4. The energy-level spectra (relative to the ground-state energy) of $SU(3)$ multiplets with $\{n^4\}$ supermultiplet symmetry, for $n = 1, \ldots, 5$, in the 2s1d shell.

The $U(4)$ Lie algebra, spanned by the operators $\{\hat{C}_{\sigma\tau}\}$ given in equation (28), has Casimir operator

$$\hat{C}_{U4} = \sum_{\sigma} \hat{C}_{\sigma\sigma}\hat{C}_{\sigma\sigma} + \sum_{\sigma\neq\tau} \hat{C}_{\sigma\tau}\hat{C}_{\tau\sigma}, \tag{67}$$

which, in an irrep with highest weight $\{X\}$ whose components are the eigenvalues of the weight operators $\{\hat{C}_{\sigma\sigma}\}$, takes the value

$$\langle\{X\}|\hat{C}_{U4}|\{X\}\rangle = \sum_{\sigma} X_\sigma(X_\sigma + 5 - 2\sigma). \tag{68}$$

The $O(8)$ Lie algebra with basis defined as in equations (26)–(28) has Casimir operator

$$\hat{C}_{O8} = \hat{C}_{U4} + \sum_{\sigma<\tau}(\hat{A}_{\sigma\tau}\hat{B}_{\sigma\tau} + \hat{B}_{\sigma\tau}\hat{A}_{\sigma\tau}), \tag{69}$$

which, for an irrep with highest weight $[Y]$ whose components are eigenvalues of the weight operators $\{\hat{C}_{\sigma\sigma}\}$, takes the value

$$\langle[Y]|\hat{C}_{O8}|[Y]\rangle = \sum_{\sigma} Y_\sigma(Y_\sigma + 8 - 2\sigma). \tag{70}$$

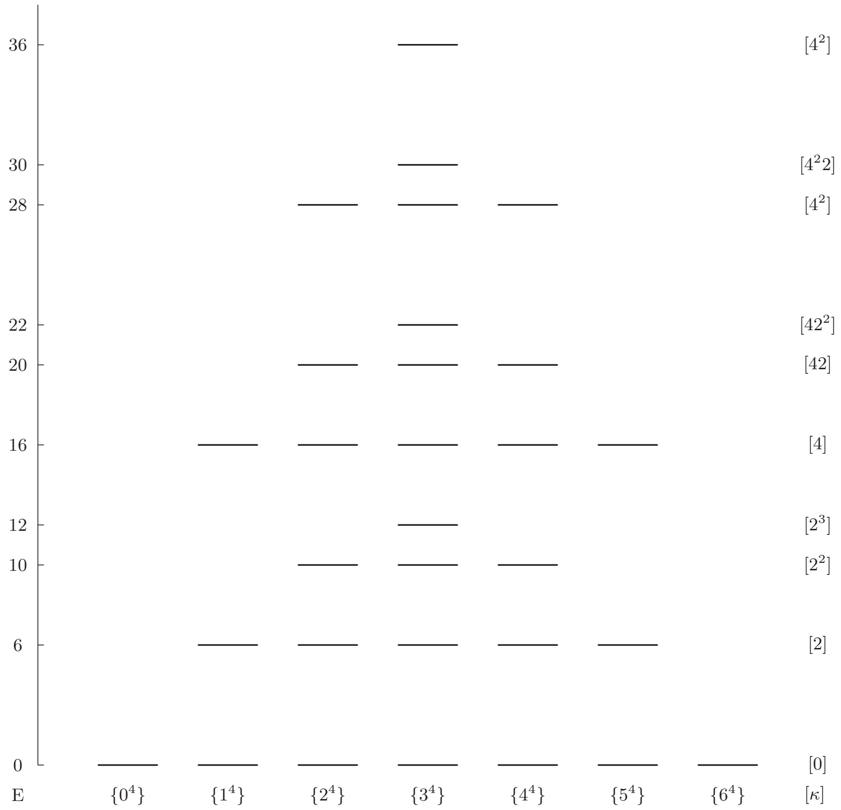


Figure 5. The energy-level spectra (relative to the ground-state energy) of the Hamiltonian $\hat{V}^P_{O6}$ for states of $\{n^4\}$ supermultiplet symmetry. Each horizontal energy level gives the degenerate excitation energy of a multiplet of states of an $O(6)$ irrep $[\kappa]$.

It follows that the $O(6)$ pairing interaction can be expressed as

$$\hat{V}^P_{O6} = \hat{\mathcal{C}}_{O8} - \hat{\mathcal{C}}_{U4} + 3\sum_{\sigma} \hat{C}_{\sigma\sigma}, \tag{71}$$

and has a spectrum given by

$$\langle\{\Lambda\}[\kappa]|\hat{V}^P_{O6}|\{\Lambda\}[\kappa]\rangle = \sum_{\sigma}[Y_\sigma(\kappa)(Y_\sigma(\kappa) + 8 - 2\sigma) - X_\sigma(\Lambda)(X_\sigma(\Lambda) + 2 - 2\sigma)]. \tag{72}$$

Now, for a $U(6)$ irrep $\{\Lambda\}$, the dual irrep $\{X(\Lambda)\}$ of the $U(4) \subset O(8)$ group with infinitesimal generators $\{\hat{\mathcal{C}}_{\sigma\tau} = \hat{C}^{(4)}_{\sigma\tau} - 3\delta_{\sigma\tau}\}$ has highest weight components

$$X_\sigma(\Lambda) = \tilde{\Lambda}_\sigma - 3. \tag{73}$$

Likewise, for an $O(6)$ irrep $[\kappa]$, the dual irrep $[3(\tilde{\kappa})]$ of $O(8)$ has highest weight $[Y(\kappa)]$ with components

$$Y_\sigma(\kappa) = 3 - \tilde{\kappa}_{5-\sigma}. \tag{74}$$

The spectrum of the Hamiltonian $\hat{H} = -\hat{V}^P_{O6}$ is shown for the states with supermultiplet symmetry $\{n^4\}$ and $n = 1, \ldots, 6$ in figure 5.

A richer spectrum could be given in the $O(6)$ basis for any Hamiltonian of the form

$$\hat{H} = -\chi_1 \hat{V}^P_{O6} - \chi_2 \hat{\mathcal{C}}_{O5} + \chi_3 \hat{L} \cdot \hat{L} \tag{75}$$

where $\hat{\mathcal{C}}_{O5}$ is the Casimir operator for $O(5)$.

3.3. The U(5) chain

The duality between the $U(5) \supset O(5)$ and $O(8) \supset U(4)$ subgroup chains also implies that the $U(5)$ coupling scheme, defined by the chain of equation (58), diagonalizes a pairing interaction $\hat{V}^P_{O5}$, now restricted to $(l = 2)L = 0$ pairs whose spectrum is derived as follows.

First observe that the spectrum of $U(5)$ irreps contained within a given $U(6)$ irrep $\{\Lambda\}$ is given by the well-known branching rule [26]

$$U(6) \downarrow U(5) \times U(1) : \{\Lambda\} \downarrow \bigoplus_{\mu m} \Gamma^{\Lambda}_{\mu m} \{\mu\} \times \{m\}, \tag{76}$$

where the $U(1)$ quantum number m is equal to the number n_s of nucleons in the $2s$ single-particle level. Thus, with Λ of maximum length 4 (so that $\tilde{\Lambda}$ is a $U(4)$ irrep)

$$U(6) \downarrow U(5) : \{\Lambda\} \downarrow \bigoplus_{m=0}^{4} \bigoplus_{\mu} \Gamma^{\Lambda}_{\mu m} \{\mu\}, \tag{77}$$

where $\Gamma^{\Lambda}_{\mu m}$ is a Littlewood–Richardson coefficient. The spectrum of $O(5)$ irreps within each $U(5)$ irrep is then determined by the $O(8) \downarrow U(4)$ branching rules for $N = 5$ given in the appendix.

Thus, for the $O(5)$ pairing interaction we obtain

$$\langle \{\mu\}[\nu] | \hat{V}^P_{O5} | \{\mu\}[\nu] \rangle = \sum_{\sigma} [Y_\sigma(\nu)(Y_\sigma(\nu) + 8 - 2\sigma) - X_\sigma(\mu)(X_\sigma(\mu) + 2 - 2\sigma)], \tag{78}$$

where

$$Y_\sigma(\nu) = \tfrac{5}{2} - \nu_{5-\sigma}, \tag{79}$$

$$X_\sigma(\mu) = \tilde{\mu}_\sigma - \tfrac{5}{2}. \tag{80}$$

The spectrum of energy levels for the Hamiltonian $\hat{H} = -\hat{V}^P_{O5}$ is shown in figures 6–8.

It is interesting to note the identical spectra shown by these figures for different nuclei. These symmetries arise because the $O(5)$-pairing Hamiltonian depends only on the Casimir invariants of $U(5)$ and its $O(5)$ subgroup which take the same values for the irreps in the corresponding nuclei. The identity between the $n_s = 0$ energies of $4n$ and the $n_s = 4$ energies of their $4(n + 1)$ neighbours is immediately understood from the symmetry of the $U(6) \downarrow U(1) \times U(5)$ branching rule of equation (76) which gives

$$\begin{aligned} U(6) \downarrow U(1) \times U(5) &: \{4^n\} \downarrow \{0\} \times \{4^n\} \oplus \cdots \\ &: \{4^{n+1}\} \downarrow \{4\} \times \{4^n\} \oplus \cdots . \end{aligned} \tag{81}$$

The other symmetries can be understood in terms of particle–hole conjugation. First observe that the fully antisymmetry irrep $\{1^5\}$ of $U(5)$ is one dimensional and corresponds to the representation of an element $g \in U(5)$ by its determinant, i.e., $g \to \varepsilon(g) = \det(g)$. Similarly, the fully closed-shell $U(5)$ irrep $\{4^5\}$ is the one-dimensional map $g \to [\varepsilon(g)]^4$. Now, a state with n-nucleons added to the 2s1d-shell vacuum state can be identified with a state of $(24 - n)$-holes added to the filled 2s1d-shell state. Thus, there is an equivalence of particle and hole representations of $U(5)$ given by

$$\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\} \equiv \varepsilon^4 \{5 - \mu_5, 5 - \mu_4, 5 - \mu_3, 5 - \mu_2, 5 - \mu_1\}. \tag{82}$$

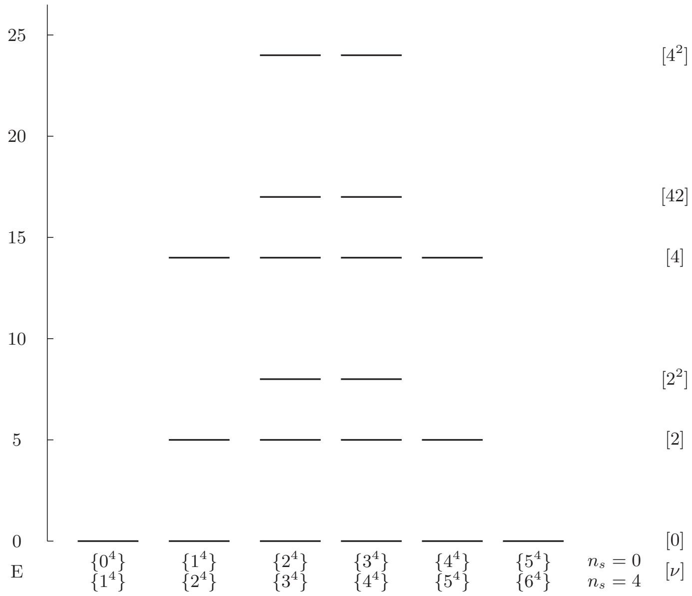


Figure 6. The energy-level spectra (relative to the ground-state energy) of $O(5)$ multiplets with $\{n^4\}$ supermultiplet symmetry, for $n_s = 0$ and $n_s = 4$. Observe that the excitation energies of the nuclei with $4n + n_s$ and $4(5 - n) + n_s$ nucleons are identical and that the $n_s = 0$ energies for $4n$ nuclei are identical to the $n_s = 4$ energies for $4(n + 1)$ nuclei.

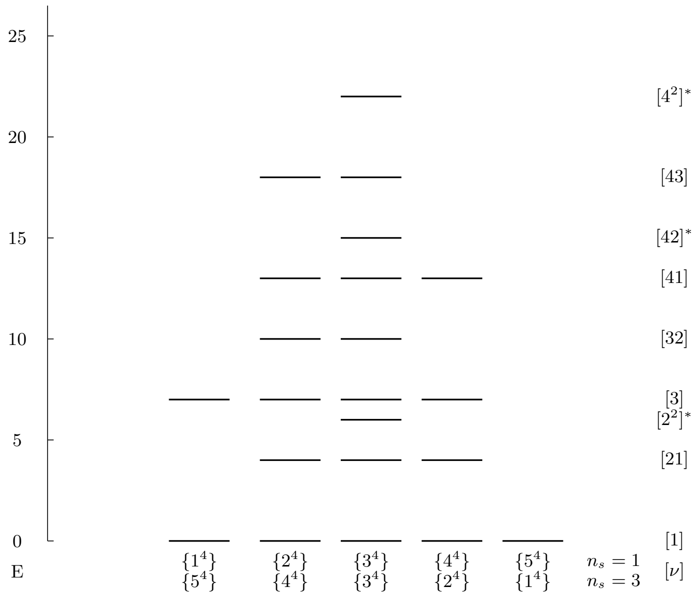


Figure 7. The energy-level spectra (relative to the ground-state energy) of $O(5)$ multiplets with $\{n^4\}$ supermultiplet symmetry, for $n_s = 1$ and $n_s = 3$. Observe that the excitation energies of the nuclei with $4n$ and $4(6 - n)$ nucleons are identical.

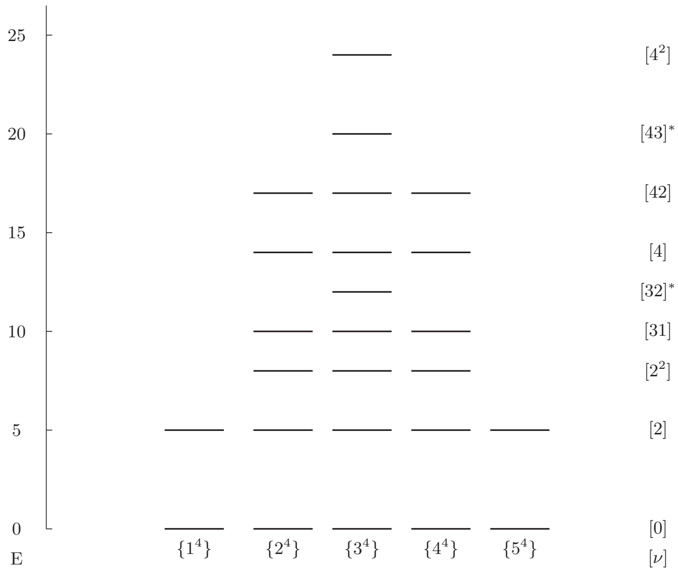


Figure 8. The energy-level spectra (relative to the ground-state energy) of $O(5)$ multiplets with $\{n^4\}$ supermultiplet symmetry, for $n_s = 2$.

For example, the $U(6) \downarrow U(1) \times U(5)$ branching rule of equation (76) gives

$$\begin{aligned} U(6) \downarrow U(1) \times U(5) &: \{4^2\} \downarrow \{1\} \times \{4, 3\} \oplus \cdots \\ &: \{4^4\} \downarrow \{3\} \times \{4^3, 1\} \oplus \cdots \end{aligned} \tag{83}$$

and particle–hole conjugation gives

$$\{4^3, 1\} \equiv \varepsilon^4 \{4, 3\}. \tag{84}$$

The factor ε^4 does not affect the relative energies of states. Hence, it follows that the spectrum of $n_s = 1$ states of $U(6)$ symmetry $\{4^2\}$ (supermultiplet symmetry $\{2^4\}$) is identical to that of the $n_s = 3$ states of $U(6)$ symmetry $\{4^4\}$ (supermultiplet symmetry $\{4^4\}$).

A richer spectrum could also be obtained in the $U(5)$ basis for any Hamiltonian of the form

$$\hat{H} = \chi_1 \hat{V}_{O5} + \chi_2 \hat{n}_s + \chi_3 (\hat{n})_s^2 + \chi_4 \hat{L} \cdot \hat{L} \tag{85}$$

where $\hat{n}_s$ is the number operator for nucleons in the 2s orbital.

4. Discussion

In this paper, we have proved a duality relationship between the irreps of the group $O(N)$, which can be used together with $U(N)$ and the $U(4)$ supermultiplet group to classify nuclear states in LST-coupling, and those of the group $O(8)$ generated by the $L = 0$ two-nucleon pair-creation operators. Although such a duality relationship has long been suspected, from the results of Bayman [10] and Flowers and Spzikowshi [7, 8], a complete proof has not, to our knowledge, been given previously, although a proof that some of the conditions necessary for a partial duality between $SO(N)$ and $SO(8)$ irreps has been given recently in [22].

As anticipated, such a duality relationship extends the well-known $SU(2)$ quasispin-symplectic and $O(5)$-symplectic dualities [6] that underlie the $J = 0$ pairing models in jj-coupling to parallel $L = 0$ pairing models in LS-coupling.

Duality of the $O(N)$ and $O(8)$ representations opens the door to tackling the longstanding problem in nuclear physics of handling the competing dynamical symmetries involved for a Hamiltonian with a mixture of a $Q \cdot Q$ and pairing interaction. The competition between a $Q \cdot Q$ and a $J = 0$ pairing interaction is complicated by the fact that the $J = 0$ pairing interaction generally involves a coupling between the spatial and spin degrees of freedom of the nucleons, whereas both a $Q \cdot Q$ and an $L = 0$ pairing interaction preserve the dynamical symmetries of the subgroup chain

$$U(4) \times U(N) \supset SU(2)_T \times SU(2)_S \times SO(3)_L \supset SU(2)_J.$$

Moreover, in an LST-coupling scheme based on harmonic-oscillator spatial states, intermediate subgroups can be inserted between $U(N)$ and its $SO(3)_L$ subgroup, which diagonalize different components of the interaction. For example, basis states which reduce the $U(N) \supset SU(3) \supset SO(3)_L$ subgroups diagonalize the $Q \cdot Q$ interaction while basis states which reduce the $U(N) \supset O(N) \supset SO(3)_L$ subgroups diagonalize an $L = 0$ pairing interaction. As shown in the last section for the 2s1d shell, other intermediate subgroups can also be invoked when $N = \sum_k (2l_k + 1)$ and the shell contains more than one single-particle angular momentum l_k. For example, if the shell contains two values of the single-particle angular momentum and $N = N_1 + N_2$ with $N_1 = 2l_1 + 1$ and $N_2 = 2l_2 + 1$, then we also have the subgroup chain

$$U(N) \supset U(N_1) \times U(N_2) \supset O(N_1) \times O(N_2) \supset SO(3)_{L_1} \times SO(3)_{L_2} \supset SO(3)_L, \qquad (86)$$

with $\mathbf{L} = \mathbf{L_1} + \mathbf{L_2}$.

Acknowledgments

The authors thank Santo D'Agostino, Joe Repka and John Wood for helpful discussions. Support of this work by grants from the Natural Sciences and Engineering Research Council of Canada is also acknowledged with appreciation.

Appendix A. Tables of $O(8) \downarrow U(4)$ branching rules for $1 \leqslant N = \sum_k (2l_k + 1) \leqslant 6$

Table A1. The even- and odd-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 2$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[1(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$			
2	$\{1^2\}$	$\{2\}$	$\{1^2\}$	
4	$\{1^4\}, \{2^2\}$	$\{21^2\}$	$\{1^4\}, \{21^2\}$	$\{1^4\}$
6	$\{2^2 1^2\}$	$\{2^3\}$	$\{2^2 1^2\}$	
8	$\{2^4\}$			
$[\tilde{\kappa}]$	$[0]$	$[2]$	$[1^2]$	$[1^4]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1	$\{1\}$	
3	$\{1^3\}, \{21\}$	$\{1^3\}$
5	$\{21^3\}, \{2^2 1\}$	$\{21^3\}$
7	$\{2^3 1\}$	
$[\tilde{\kappa}]$	$[1]$	$[1^3]$

Table A2. The even- and odd-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 3$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $\left[\frac{3}{2}(\tilde{\kappa})\right]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$				
2	$\{1^2\}$	$\{2\}$	$\{1^2\}$		
4	$\{1^4\}, \{2^2\}$	$\{21^2\}, \{31\}$	$\{1^4\}, \{21^2\}, \{2^2\}$	$\{21^2\}$	$\{1^4\}$
6	$\{2^21^2\}, \{3^2\}$	$\{31^3\}, \{2^3\}, \{321\}$	$\{2^21^2\}^2, \{321\}$	$\{2^21^2\}, \{31^3\}, \{2^3\}$	$\{2^21^2\}$
8	$\{2^4\}, \{3^21^2\}$	$\{32^21\}, \{3^22\}$	$\{2^4\}, \{32^21\}, \{3^21^2\}$	$\{32^21\}$	$\{2^4\}$
10	$\{3^22^2\}$	$\{3^31\}$	$\{3^22^2\}$		
12	$\{3^4\}$				
$[\tilde{\kappa}]$	$[0]$	$[2]$	$[1^2]$	$[21^2]$	$[1^4]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1		$\{1\}$			
3	$\{3\}$	$\{1^3\}, \{21\}$	$\{21\}$	$\{1^3\}$	
5	$\{31^2\}$	$\{21^3\}, \{2^21\}, \{32\}$	$\{21^3\}, \{2^21\}, \{31^2\}$	$\{21^3\}, \{2^21\}$	$\{21^3\}$
7	$\{32^2\}$	$\{2^31\}, \{321^2\}, \{3^21\}$	$\{2^31\}\{321^2\}, \{32^2\}$	$\{2^31\}, \{321^2\}$	$\{2^31\}$
9	$\{3^3\}$	$\{32^3\}, \{3^221\}$	$\{3^221\}$	$\{32^3\}$	
11		$\{3^32\}$			
$[\tilde{\kappa}]$	$[3]$	$[1]$	$[21]$	$[1^3]$	$[21^3]$

Table A3. The even-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 4$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[2(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$				
2	$\{1^2\}$		$\{2\}$	$\{1^2\}$	
4	$\{1^4\}, \{2^2\}$	$\{4\}$	$\{21^2\}, \{31\}$	$\{1^4\}, \{21^2\}, \{2^2\}$	$\{31\}$
6	$\{2^21^2\}, \{3^2\}$	$\{41^2\}$	$\{31^3\}, \{2^3\}, \{321\}, \{42\}$	$\{2^21^2\}^2, \{3^2\}, \{321\}$	$\{31^3\}, \{41^2\}, \{321\}$
8	$\{4^2\}, \{3^21^2\}, \{2^4\}$	$\{42^2\}$	$\{3^22\}, \{32^21\}, \{421^2\}, \{431\}$	$\{2^4\}, \{3^21^2\}^2, \{431\}, \{32^21\}$	$\{3^22\}, \{421^2\}, \{42^2\}\{32^21\}$
10	$\{3^22^2\}, \{4^21^2\}$	$\{43^2\}$	$\{3^31\}, \{42^3\}, \{4321\}, \{4^22\}$	$\{3^22^2\}^2, \{4^21^2\}, \{4321\}$	$\{3^31\}, \{43^2\}, \{4321\}$
12	$\{3^4\}, \{4^22^2\}$	$\{4^3\}$	$\{43^22\}, \{4^231\}$	$\{3^4\}, \{43^22\}, \{4^22^2\}$	$\{4^231\}$
14	$\{4^23^2\}$		$\{4^32\}$	$\{4^23^2\}$	
16	$\{4^4\}$				
$[\tilde{\kappa}]$	$[0]$	$[4]$	$[2]$	$[1^2]$	$[31]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0							
2							
4	$\{21^2\}$	$\{2^2\}$	$\{1^4\}$				
6	$\{2^21^2\}, \{31^3\}, \{2^3\}, \{321\}$	$\{2^21^2\}, \{321\}$	$\{2^21^2\}$	$\{31^3\}$	$\{2^3\}$	$\{2^21^2\}$	
8	$\{421^2\}, \{3^22\}, \{3^21^2\}, \{32^21\}^2$	$\{42^2\}, \{3^21^2\}, \{32^21\}, \{2^4\}$	$\{2^4\}, \{3^21^2\}$	$\{32^21\}$	$\{32^21\}$	$\{2^4\}, \{32^21\}$	$\{2^4\}$
10	$\{3^22^2\}, \{3^31\}, \{42^3\}, \{4321\}$	$\{3^22^2\}, \{4321\}$	$\{3^22^2\}$	$\{3^31\}$	$\{42^3\}$	$\{3^22\}$	
12	$\{43^22\}$	$\{4^22^2\}$	$\{3^4\}$				
14							
16							
$[\tilde{\kappa}]$	$[21^2]$	$[2^2]$	$[1^4]$	$[31^3]$	$[2^3]$	$[2^21^2]$	$[2^4]$

Table A4. The odd-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 4$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[2(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1	$\{1\}$		
3	$\{1^3\}, \{21\}$	$\{3\}$	$\{21\}$
5	$\{32\}, \{2^21\}, \{21^3\}$	$\{41\}, \{31^2\}$	$\{32\}, \{31^2\}, \{2^21\}, \{21^3\}$
7	$\{43\}, \{3^21\}, \{321^2\}, \{2^31\}$	$\{421\}, \{32^2\}, \{41^3\}$	$\{421\}, \{3^21\}, \{32^2\}, \{321^2\}^2, \{2^31\}$
9	$\{4^21\}, \{431^2\}, \{3^221\}, \{32^3\}$	$\{432\}, \{42^21\}, \{3^3\}$	$\{432\}, \{431^2\}, \{42^21\}, \{3^221\}^2, \{32^3\}$
11	$\{4^221\}, \{432^2\}, \{3^32\}$	$\{4^23\}, \{43^21\}$	$\{4^221\}, \{43^21\}, \{432^2\}, \{3^32\}$
13	$\{43^3\}, \{4^232\}$	$\{4^31\}$	$\{4^232\}$
15	$\{4^33\}$		
$[\tilde{\kappa}]$	$[1]$	$[3]$	$[21]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1					
3	$\{1^3\}$				
5	$\{2^21\}, \{21^3\}$	$\{31^2\}$	$\{2^21\}$	$\{21^3\}$	
7	$\{3^21\}, \{321^2\}, \{2^31\}$	$\{32^2\}, \{41^3\}, \{321^2\}$	$\{32^2\}, \{321^2\}, \{2^31\}$	$\{321^2\}, \{2^31\}$	$\{2^31\}$
9	$\{431^2\}, \{3^221\}, \{32^3\}$	$\{42^21\}, \{3^3\}, \{3^221\}$	$\{42^21\}, \{3^221\}, \{32^3\}$	$\{3^221\}, \{32^3\}$	$\{32^3\}$
11	$\{432^2\}, \{3^32\}$	$\{43^21\}$	$\{432^2\}$	$\{3^32\}$	
13	$\{43^3\}$				
15					
$[\tilde{\kappa}]$	$[1^3]$	$[31^2]$	$[2^21]$	$[21^3]$	$[2^31]$

Table A5. The even-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 5$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[\frac{5}{2}(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$			
2	$\{1^2\}$	$\{1^2\}$	$\{2\}$	
4	$\{2^2\}, \{1^4\}$	$\{2^2\}, \{21^2\}, \{1^4\}$	$\{31\}, \{21^2\}$	$\{4\}$
6	$\{3^2\}, \{2^21^2\}$	$\{3^2\}, \{321\}, \{2^21^2\}^2$	$\{42\}, \{321\}, \{2^3\}, \{31^3\}$	$\{51\}, \{41^2\}$
8	$\{4^2\}, \{3^21^2\}, \{2^4\}$	$\{4^2\}, \{431\}, \{3^21^2\}^2, \{32^21\}, \{2^4\}$	$\{53\}, \{431\}, \{3^22\}, \{421^2\}, \{32^21\}$	$\{521\}, \{42^2\}, \{51^3\}$
10	$\{5^2\}, \{4^21^2\}, \{3^22^2\}$	$\{541\}, \{4^21^2\}^2, \{4321\}, \{3^22^2\}^2$	$\{541\}, \{531^2\}, \{4^22\}, \{4321\}, \{42^3\}, \{3^31\}$	$\{532\}, \{52^21\}, \{43^2\}$
12	$\{5^21^2\}, \{4^22^2\}, \{3^4\}$	$\{5^21^2\}, \{5421\}, \{4^22^2\}^2, \{43^22\}, \{3^4\}$	$\{5^22\}, \{5421\}, \{532^2\}, \{4^231\}, \{43^22\}$	$\{543\}, \{53^21\}, \{4^3\}$
14	$\{5^22^2\}, \{4^23^2\}$	$\{5^22^2\}, \{5432, \{4^23^2\}^2$	$\{5^231\}, \{5432\}, \{53^3\}, \{4^32\}, \{4^231\}$	$\{5^24\}, \{54^21\}$
16	$\{5^23^2\}, \{4^4\}$	$\{5^23^2\}, \{54^23\}, \{4^4\}$	$\{54^2\}, \{54^23\}$	$\{5^31\}$
18	$\{5^24^2\}$	$\{5^24^2\}$	$\{5^33\}$	
20	$\{5^4\}$			
$[\tilde{\kappa}]$	$[0]$	$[1^2]$	$[2]$	$[4]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0		
2		
4	$\{31\}$	$\{2^2\}$
6	$\{42\}, \{41^2\}, \{321\}, \{31^3\}$	$\{3^2\}, \{321\}, \{2^21^2\}$
8	$\{521\}, \{431\}, \{42^2\}, \{3^22\}, \{421^2\}^2, \{32^21\}$	$\{431\}, \{42^2\}, \{3^21^2\}^2, \{32^21\}, \{2^4\}$
10	$\{532\}, \{4^22\}, \{43^2\}, \{531^2\}, \{52^21\}, \{4321\}, \{3^31\}, \{42^3\}$	$\{532\}, \{4^21^2\}, \{4321\}^2, \{3^22^2\}^2$
12	$\{543\}, \{5421\}, \{53^21\}, \{532^2\}, \{4^231\}, \{43^22\}$	$\{5421\}, \{53^21\}, \{4^22^2\}^2, \{43^22\}, \{3^4\}$
14	$\{5^231\}, \{54^21\}, \{5432\}, \{4^32\}$	$\{5^22^2\}, \{5432\}, \{4^23^2\}$
16	$\{5^242\}$	$\{5^23^2\}$
18		
20		
$[\tilde{\kappa}]$	$[31]$	$[2^2]$

n	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$
4	$\{21^2\}$	$\{1^4\}$
6	$\{321\}, \{2^3\}, \{31^3\}, \{2^21^2\}$	$\{2^21^2\}$
8	$\{431\}, \{3^22\}, \{421^2\}, \{3^21^2\}, \{32^21\}^2$	$\{3^31^2\}, \{2^4\}$
10	$\{4^22\}, \{531^2\}, \{4^21^2\}, \{4321\}^2, \{3^31\}, \{42^3\}, \{3^22^2\}$	$\{4^21^2\}, \{3^22^2\}$
12	$\{5421\}, \{532^2\}, \{4^231\}, \{4^22^2\}, \{43^22\}^2$	$\{4^22^2\}, \{3^4\}$
14	$\{5421\}, \{532^2\}, \{4^32\}, \{4^23^2\}$	$\{4^23^2\}$
16	$\{54^23\}$	$\{4^4\}$
$[\tilde\kappa]$	$[21^2]$	$[1^4]$

n	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$
0				
2				
4				
6	$\{41^2\}$	$\{321\}$	$\{2^3\}$	$\{31^3\}$
8	$\{42^2\}, \{51^3\}, \{421^2\}$	$\{42^2\}, \{3^22\}, \{3^21^2\}, \{32^21\}, \{421^2\}$	$\{3^22\}, \{32^21\}$	$\{421^2\}, \{32^21\}$
10	$\{43^2\}, \{52^21\}, \{4321\}$	$\{43^2\}, \{52^21\}, \{4321\}^2, \{3^31\}, \{42^3\}, \{3^22^2\}$	$\{4321\}, \{3^31\}, \{42^3\}$	$\{4321\}, \{3^31\}, \{42^3\}$
12	$\{53^21\}, \{4^3\}, \{4^231\}$	$\{53^21\}, \{532^2\}, \{4^22^2\}, \{43^22\}, \{4^231\}$	$\{532^2\}, \{43^22\}$	$\{4^231\}, \{43^22\}$
14	$\{54^21\}$	$\{5432\}$	$\{53^3\}$	$\{4^32\}$
16				
18				
20				
$[\tilde\kappa]$	$[41^2]$	$[321]$	$[2^3]$	$[31^3]$

n	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$
0			
2			
4			
6	$\{2^21^2\}$		
8	$\{3^21^2\}, \{32^21\}, \{2^4\}$	$\{32^21\}$	$\{2^4\}$
10	$\{4321\}, \{3^22^2\}^2$	$\{3^31\}, \{42^3\}, \{3^22^2\}$	$\{3^22^2\}$
12	$\{4^22^2\}, \{43^22\}, \{3^4\}$	$\{43^22\}$	$\{3^4\}$
14	$\{4^23^2\}$		
16			
18			
20			
$[\tilde\kappa]$	$[2^21^2]$	$[32^21]$	$[2^4]$

Table A6. The odd-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 5$. The $U(4)$ irreps are labelled by $\{\tilde\lambda\}$ and the $O(8)$ irreps $[\frac{5}{2}(\tilde\kappa)]$ simply by $[\tilde\kappa]$.

n	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$	$\{\tilde\lambda\}$
1	$\{1\}$		
3	$\{21\}, \{1^3\}$	$\{3\}$	$\{1^3\}$
5	$\{32\}, \{2^21\}, \{21^3\}$	$\{41\}, \{31^2\}$	$\{2^21\}, \{21^3\}$
7	$\{43\}, \{3^21\}, \{321^2\}, \{2^31\}$	$\{52\}, \{421\}, \{32^2\}, \{41^3\}$	$\{3^21\}, \{321^2\}, \{2^31\}$
9	$\{54\}, \{4^21\}, \{431^2\}, \{3^221\}, \{32^3\}$	$\{531\}, \{432\}, \{3^3\}, \{521^2\}, \{42^21\}$	$\{4^21\}, \{431^2\}, \{3^221\}, \{32^3\}$
11	$\{5^21\}, \{541^2\}, \{4^221\}, \{432^2\}, \{3^32\}$	$\{542\}, \{5321\}, \{52^3\}, \{4^23\}, \{43^21\}$	$\{541^2\}, \{4^221\}, \{432^2\}, \{3^32\}$
13	$\{5^221\}, \{542^2\}, \{4^232\}, \{43^3\}$	$\{5^23\}, \{5431\}, \{53^22\}, \{4^31\}$	$\{542^2\}, \{4^232\}, \{43^3\}$
15	$\{5^232\}, \{543^2\}, \{4^33\}$	$\{5^241\}, \{54^22\}$	$\{543^2\}, \{4^33\}$
17	$\{5^243\}, \{54^3\}$	$\{5^32\}$	$\{54^3\}$
19	$\{5^34\}$		
$[\tilde\kappa]$	$[1]$	$[3]$	$[1^3]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1			
3	$\{21\}$		
5	$\{31^2\}, \{32\}, \{2^21\}, \{21^3\}$	$\{5\}$	$\{41\}$
7	$\{43\}, \{3^21\}, \{421\}, \{32^2\}, \{321^2\}^2, \{2^31\}$	$\{51^2\}$	$\{51^2\}, \{421\}, \{41^3\}$
9	$\{531\}, \{4^21\}, \{432\}, \{431^2\}^2, \{42^21\}, \{3^221\}^2, \{32^3\}$	$\{52^2\}$	$\{52^2\}, \{432\}, \{521^2\}, \{42^21\}$
11	$\{542\}, \{541^2\}, \{5321\}, \{4^221\}^2, \{43^21\}, \{432^2\}^2, \{3^32\}$	$\{53^2\}$	$\{53^2\}, \{5321\}, \{4^23\}, \{43^21\}$
13	$\{5^221\}, \{542^2\}, \{5431\}, \{53^22\}, \{4^232\}^2, \{43^3\}$	$\{54^2\}$	$\{54^2\}, \{5431\}, \{4^31\}$
15	$\{54^22\}, \{5^232\}, \{543^2\}, \{4^33\}$	$\{5^3\}$	$\{5^241\}$
17	$\{5^243\}$		
19			
$[\tilde{\kappa}]$	$[21]$	$[5]$	$[41]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1		
3		
5	$\{32\}$	$\{31^2\}$
7	$\{421\}, \{3^21\}, \{321^2\}$	$\{421\}, \{32^2\}, \{41^3\}, \{321^2\}$
9	$\{52^2\}, \{432\}, \{431^2\}, \{42^21\}, \{3^221\}, \{32^3\}$	$\{432\}, \{3^3\}, \{521^2\}, \{431^2\}, \{42^21\}^2, \{3^221\}$
11	$\{53^2\}, \{5321\}, \{4^221\}, \{43^21\}, \{432^2\}, \{3^32\}$	$\{5321\}, \{52^3\}, \{4^23\}, \{4^221\}, \{43^21\}^2, \{432^2\}$
13	$\{5431\}, \{542^2\}, \{4^232\}$	$\{5431\}, \{53^22\}, \{4^31\}, \{4^232\}$
15	$\{5^232\}$	$\{54^22\}$
17		
19		
$[\tilde{\kappa}]$	$[32]$	$[31^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1		
3		
5	$\{2^21\}$	$\{21^3\}$
7	$\{3^21\}, \{32^2\}, \{321^2\}, \{2^31\}$	$\{321^2\}, \{2^31\}$
9	$\{432\}, \{431^2\}, \{42^21\}, \{3^221\}^2, \{32^3\}$	$\{431^2\}, \{3^221\}, \{32^3\}$
11	$\{5321\}, \{4^221\}, \{43^21\}, \{432^2\}^2, \{3^32\}$	$\{4^221\}, \{432^2\}, \{3^32\}$
13	$\{542^2\}, \{53^22\}, \{4^232\}, \{43^3\}$	$\{4^232\}, \{43^3\}$
15	$\{543^2\}$	$\{4^33\}$
17		
19		
$[\tilde{\kappa}]$	$[2^21]$	$[21^3]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1					
3					
5					
7	$\{32^2\}$	$\{41^3\}$	$\{321^2\}$	$\{2^31\}$	
9	$\{3^3\}, \{42^21\}, \{3^221\}$	$\{42^21\}$	$\{42^21\}, \{3^221\}, \{32^3\}$	$\{3^221\}, \{32^3\}$	$\{32^3\}$
11	$\{52^3\}, \{43^21\}, \{432^2\}$	$\{43^21\}$	$\{43^21\}, \{432^2\}, \{3^32\}$	$\{432^2\}, \{3^32\}$	$\{3^32\}$
13	$\{53^22\}$	$\{4^31\}$	$\{4^232\}$	$\{43^3\}$	
15					
17					
19					
$[\tilde{\kappa}]$	$[32^2]$	$[41^3]$	$[2^31]$	$[3^32]$	$[32^3]$

Table A7. The even-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 6$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[3(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
0	$\{0\}$			
2	$\{1^2\}$		$\{1^2\}$	
4	$\{2^2\}, \{1^4\}$		$\{2^2\}, \{21^2\}, \{1^4\}$	
6	$\{3^2\}, \{2^2 1^2\}$	$\{6\}$	$\{3^2\}, \{321\}, \{2^2 1^2\}^2$	$\{51\}$
8	$\{4^2\}, \{3^2 1^2\}, \{2^4\}$	$\{61^2\}$	$\{4^2\}, \{431\}, \{3^2 1^2\}^2, \{32^2 1\}, \{2^4\}$	$\{61^2\}, \{521\}, \{51^3\}$
10	$\{5^2\}, \{4^2 1^2\}, \{3^2 2^2\}$	$\{62^2\}$	$\{5^2\}, \{541\}, \{4^2 1^2\}^2, \{4321\}, \{3^2 2^2\}^2$	$\{62^2\}, \{532\}, \{621^2\}, \{52^2 1\}$
12	$\{6^2\}, \{5^2 1^2\}, \{4^2 2^2\}, \{3^4\}$	$\{63^2\}$	$\{651\}, \{5^2 1^2\}^2, \{5421\}, \{4^2 2^2\}^2, \{43^2 2\}, \{3^4\}$	$\{63^2\}, \{6321\}, \{543\}, \{53^2 1\}$
14	$\{6^2 1^2\}, \{5^2 2^2\}, \{4^2 3^2\}$	$\{64^2\}$	$\{6^2 1^2\}, \{6521\}, \{5^2 2^2\}^2, \{5432\}, \{4^2 3^2\}^2$	$\{64^2\}, \{6431\}, \{5^2 4\}, \{54^2 1\}$
16	$\{6^2 2^2\}, \{5^2 3^2\}, \{4^4\}$	$\{65^2\}$	$\{6^2 2^2\}, \{6532\}, \{5^2 3^2\}^2, \{54^2 3\}, \{4^4\}$	$\{65^2\}, \{6541\}, \{5^3 1\}$
18	$\{6^2 3^3\}, \{5^2 4^2\}$	$\{6^3\}$	$\{6^2 3^3\}, \{6543\}, \{5^2 4^2\}^2$	$\{6^2 51\}$
20	$\{6^2 4^2\}, \{5^4\}$		$\{6^2 4^2\}, \{5^4\}$	
22	$\{6^2 5^2\}$		$\{6^2 5^2\}$	
24	$\{6^4\}$			
$[\tilde{\kappa}]$	$[0]$	$[6]$	$[1^2]$	$[51]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
2	$\{2\}$	
4	$\{31\}, \{21^2\}$	$\{4\}$
6	$\{42\}, \{321\}, \{2^3\}, \{31^3\}$	$\{51\}, \{41^2\}$
8	$\{53\}, \{431\}, \{3^2 2\}, \{421^2\}, \{32^2 1\}$	$\{62\}, \{521\}, \{42^2\}, \{51^3\}$
10	$\{64\}, \{541\}, \{4^2 2\}, \{531^2\}, \{4321\}, \{3^3 1\}, \{42^3\}$	$\{631\}, \{532\}, \{43^2\}, \{621^2\}, \{52^2 1\}$
12	$\{651\}, \{641^2\}, \{5^2 2\}, \{5421\}, \{532^2\}, \{4^2 31\}, \{43^2 2\}$	$\{642\}, \{6321\}, \{543\}, \{4^3\}, \{53^2 1\}, \{62^3\}$
14	$\{6^2 2\}, \{6521\}, \{642^2\}, \{5^2 31\}, \{5432\}, \{53^3\}, \{4^3 2\}$	$\{653\}, \{6431\}, \{63^2 2\}, \{5^2 4\}, \{54^2 1\}$
16	$\{6^2 31\}, \{6532\}, \{643^2\}, \{5^2 42\}, \{54^2 3\}$	$\{6^2 4\}, \{6541\}, \{64^2 2\}, \{5^3 1\}$
18	$\{6^2 42\}, \{6543\}, \{64^3\}, \{5^3 3\}$	$\{6^2 51\}, \{65^2 2\}$
20	$\{6^2 53\}, \{65^2 4\}$	$\{6^3 2\}$
22	$\{6^3 4\}$	
$[\tilde{\kappa}]$	$[2]$	$[4]$

n	$\{\tilde{\lambda}\}$
4	$\{31\}$
6	$\{42\}, \{41^2\}, \{321\}, \{31^3\}$
8	$\{53\}, \{521\}, \{431\}, \{42^2\}, \{3^2 2\}, \{421^2\}^2, \{32^2 1\}$
10	$\{631\}, \{541\}, \{532\}, \{4^2 2\}, \{43^2\}, \{531^2\}^2, \{52^2 1\}, \{3^3 1\}, \{42^3\}, \{4321\}^2$
12	$\{642\}, \{641^2\}, \{5^2 2\}, \{53^2 1\}, \{532^2\}^2, \{4^2 31\}^2, \{43^2 2\}, \{6321\}, \{543\}, \{5421\}^2$
14	$\{653\}, \{6521\}, \{6431\}, \{642^2\}, \{63^2 2\}, \{5^2 31\}^2, \{54^2 1\}, \{53^3\}, \{4^3 2\}, \{5432\}^2$
16	$\{6^2 31\}, \{6541\}, \{6532\}, \{64^2 2\}, \{643^2\}, \{5^2 42\}^2, \{54^2 3\}$
18	$\{6^2 42\}, \{65^2 2\}, \{6543\}, \{5^3 3\}$
20	$\{6^2 53\}$
$[\tilde{\kappa}]$	$[31]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
4	$\{2^2\}$	
6	$\{3^2\}, \{321\}, \{2^2 1^2\}$	$\{42\}$
8	$\{4^2\}, \{431\}, \{42^2\}, \{3^2 1^2\}^2, \{32^2 1\}, \{2^4\}$	$\{521\}, \{431\} \cdot \{421^2\}$
10	$\{541\}, \{532\}, \{4^2 1^2\}^2, \{4321\}^2, \{3^2 2^2\}^2$	$\{62^2\}, \{52^2 1\}, \{4^2 2\}, \{532\}, \{531^2\}, \{4321\}, \{42^3\}$
12	$\{642\}, \{5^2 1^2\}, \{5421\}^2, \{53^2 1\}, \{4^2 2^2\}^3, \{43^2 2\}, \{3^4\}$	$\{63^2\}, \{6321\}, \{543\}, \{5421\}, \{53^2 1\}, \{4^2 31\}, \{532^2\}, \{43^2 2\}$
14	$\{6521\}, \{6431\}, \{5^2 2^2\}^2, \{5432\}^2, \{4^2 3^2\}^2$	$\{64^2\}, \{54^2 1\}, \{642^2\}, \{6431\}, \{5^2 31\}, \{5432\}, \{4^3 2\}$
16	$\{6^2 2^2\}, \{6532\}, \{64^2 2\}, \{5^2 3^2\}^2, \{4^4\}$	$\{6541\}, \{6532\}, \{5^2 42\}, \{54^2 3\}$
18	$\{6^2 3^2\}, \{6543\}, \{5^2 4^2\}$	$\{6^2 42\}$
20	$\{6^2 4^2\}$	
$[\tilde{\kappa}]$	$[2^2]$	$[42]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
4	$\{1^4\}$	
6	$\{2^2 1^2\}$	
8	$\{3^2 1^2\}, \{2^4\}$	$\{51^3\}$
10	$\{4^2 1^2\}, \{3^2 2^2\}$	$\{52^2 1\}$
12	$\{5^2 1^2\}, \{4^2 2^2\}, \{3^4\}$	$\{53^2 1\}$
14	$\{5^2 2^2\}, \{4^2 3^2\}$	$\{54^2 1\}$
16	$\{5^2 3^2\}, \{4^4\}$	$\{5^3 1\}$
18	$\{5^2 4^2\}$	
20	$\{5^4\}$	
$[\tilde{\kappa}]$	$[1^4]$	$[51^3]$

n	$\{\tilde{\lambda}\}$
6	$\{321\}$
8	$\{431\}, \{42^2\}, \{3^2 2\}, \{421^2\}, \{3^2 1^2\}, \{32^2 1\}$
10	$\{532\}, \{4^2 2\}, \{43^2\}, \{531^2\}, \{4^2 1^2\}, \{52^2 1\}, \{4321\}^3, \{3^3 1\}, \{42^3\}, \{3^2 2^2\}$
12	$\{6321\}, \{543\}, \{5421\}^2, \{53^2 1\}^2, \{532^2\}^2, \{4^2 31\}^2, \{4^2 2^2\}^2, \{43^2 2\}^2$
14	$\{6431\}, \{642^2\}, \{63^2 2\}, \{5^2 31\}, \{5^2 2^2\}, \{54^2 1\}, \{5432\}^3, \{53^3\}, \{4^3 2\}, \{4^2 3^2\}$
16	$\{6532\}, \{64^2 2\}, \{643^2\}, \{5^2 42\}, \{5^2 3^2\}, \{54^2 3\}$
18	$\{6543\}$
$[\tilde{\kappa}]$	$[321]$

n	$\{\tilde{\lambda}\}$
4	$\{21^2\}$
6	$\{321\}, \{2^3\}, \{31^3\}, \{2^2 1^2\}$
8	$\{431\}, \{3^2 2\}, \{421^2\}, \{3^2 1^2\}, \{32^2 1\}^2$
10	$\{541\}, \{4^2 2\}, \{531^2\}, \{4^2 1^2\}, \{4321\}^2, \{3^3 1\}, \{42^3\}, \{3^2 2^2\}$
12	$\{5^2 2\}, \{641^2\}, \{5^2 1^2\}, \{5421\}^2, \{4^2 31\}, \{532^2\}, \{4^2 2^2\}, \{43^2 2\}^2$
14	$\{6521\}, \{642^2\}, \{5^2 31\}, \{5^2 2^2\}, \{5432\}^2\{53^3\}, \{4^3 2\}, \{4^2 3^2\}$
16	$\{6532\}, \{643^2\}, \{5^2 42\}, \{5^2 3^2\}, \{54^2 3\}^2$
18	$\{6543\}, \{64^3\}, \{5^3 3\}, \{5^2 4^2\}$
20	$\{65^2 4\}$
$[\tilde{\kappa}]$	$[21^2]$

n	$\{\tilde{\lambda}\}$
4	
6	$\{41^2\}$
8	$\{521\}, \{42^2\}, \{51^3\}, \{421^2\}$
10	$\{532\}, \{43^2\}, \{621^2\}, \{531^2\}, \{52^2 1\}^2, \{4321\}$
12	$\{6321\}, \{543\}, \{4^3\}, \{5421\}, \{53^2 1\}^2, \{4^2 31\}, \{532^2\}, \{62^3\}$
14	$\{6431\}, \{63^2 2\}, \{5^2 4\}, \{5^2 31\}, \{54^2 1\}^2, \{5432\}$
16	$\{6541\}\{64^2 2\}\{5^3 1\}, \{5^2 42\}$
18	$\{65^2 2\}$
20	
$[\tilde{\kappa}]$	$[41^2]$

n	$\{\tilde{\lambda}\}$
6	$\{3^2\}$
8	$\{431\}, \{3^2 1^2\}$
10	$\{532\}, \{4^2 1^2\}, \{4321\}, \{3^2 2^2\}$
12	$\{63^2\}, \{5421\}, \{53^2 1\}, \{4^2 2^2\}, \{43^2 2\}, \{3^4\}$
14	$\{6431\}, \{5^2 2^2\}, \{5432\}, \{4^2 3^2\}$
16	$\{6532\}, \{5^2 3^2\}$
18	$\{6^2 3^2\}$
$[\tilde{\kappa}]$	$[3^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
6	$\{2^3\}$	
8	$\{3^2 2\}, \{32^2 1\}$	$\{42^2\}$
10	$\{4^2 2\}, \{4321\}, \{3^3 1\}, \{42^3\}$	$\{43^2\}, \{52^2 1\}, \{4321\}$
12	$\{5421\}, \{532^2\}, \{4^2 31\}, \{43^2 2\}$	$\{62^3\}, \{4^3\}, \{53^2 1\}, \{532^2\}, \{4^2 31\}, \{4^2 2^2\}$
14	$\{642^2\}, \{5432\}, \{53^3\}, \{4^3 2\}$	$\{63^2 2\}, \{54^2 1\}, \{5432\}$
16	$\{643^2\}, \{54^2 3\}$	$\{64^2 2\}$
18	$\{64^3\}$	
$[\tilde{\kappa}]$	$[2^3]$	$[42^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
6	$\{2^2 1^2\}$	$\{31^3\}$	
8	$\{3^2 1^2\}, \{32^2 1\}, \{2^4\}$	$\{421^2\}, \{32^2 1\}$	$\{421^2\}$
10	$\{4^2 1^2\}, \{4321\}, \{3^2 2^2\}^2$	$\{531^2\}, \{4321\}, \{42^3\}, \{3^3 1\}$	$\{52^2 1\}, \{4321\}, \{42^3\}$
12	$\{5421\}, \{4^2 2^2\}^2, \{43^2 2\}, \{3^4\}$	$\{5421\}, \{532^2\}, \{4^2 31\}, \{43^2 2\}$	$\{53^2 1\}, \{532^2\}, \{4^2 31\}, \{43^2 2\}$
14	$\{5^2 2^2\}, \{5432\}, \{4^2 3^2\}^2$	$\{5^2 31\}, \{5432\}, \{4^3 2\}, \{53^3\}$	$\{54^2 1\}, \{5432\}, \{4^3 2\}$
16	$\{5^2 3^2\}, \{54^2 3\}, \{4^4\}$	$\{5^2 42\}, \{54^2 3\}$	$\{5^2 42\}$
18	$\{5^2 4^2\}$	$\{5^3 3\}$	
$[\tilde{\kappa}]$	$[2^2 1^2]$	$[31^3]$	$[421^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
8	$\{3^2 2\}$	$\{3^2 1^2\}$	$\{32^2 1\}$	$\{2^4\}$	
10	$\{43^2\}, \{4321\}, \{3^3 1\}$	$\{4321\}, \{3^2 2^2\}$	$\{4321\}, \{3^3 1\}, \{42^3\}, \{3^2 2^2\}$	$\{3^2 2^2\}$	$\{42^3\}$
12	$\{53^2 1\}, \{4^2 31\}, \{532^2\}, \{43^2 2\}$	$\{53^2 1\}, \{4^2 2^2\}, \{43^2 2\}, \{3^4\}$	$\{4^2 31\}, \{532^2\}, \{4^2 2^2\}, \{43^2 2\}^2$	$\{4^2 2^2\}, \{3^4\}$	$\{43^2 2\}$
14	$\{63^2 2\}, \{5432\}, \{53^3\}$	$\{5432\}, \{4^2 3^2\}$	$\{5432\}, \{53^3\}, \{4^3 2\}, \{4^2 3^2\}$	$\{4^2 3^2\}$	$\{4^3 2\}$
16	$\{643^2\}$	$\{5^2 3^2\}$	$\{54^2 3\}$	$\{4^4\}$	
$[\tilde{\kappa}]$	$[3^2 2]$	$[3^2 1^2]$	$[32^2 1]$	$[2^4]$	$[42^3]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
8			
10	$\{3^3 1\}$	$\{3^2 2^2\}$	
12	$\{43^2 2\}$	$\{43^2 2\}, \{3^4\}$	$\{3^4\}$
14	$\{53^3\}$	$\{4^2 3^2\}$	
16			
$[\tilde{\kappa}]$	$[3^3 1]$	$[3^2 2^2]$	$[3^4]$

Table A8. The odd-n $U(4)$ irreps contained in the $O(8)$ irreps for $N = 6$. The $U(4)$ irreps are labelled by $\{\tilde{\lambda}\}$ and the $O(8)$ irreps $[3(\tilde{\kappa})]$ simply by $[\tilde{\kappa}]$.

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1	$\{1\}$	
3	$\{1^3\}, \{21\}$	
5	$\{32\}, \{2^21\}, \{21^3\}$	$\{5\}$
7	$\{43\}, \{3^21\}, \{321^2\}, \{2^31\}$	$\{61\}, \{51^2\}$
9	$\{54\}, \{4^21\}, \{431^2\}, \{3^221\}, \{32^3\}$	$\{621\}, \{52^2\}, \{61^3\}$
11	$\{65\}, \{5^21\}, \{541^2\}, \{4^221\}, \{432^2\}, \{3^32\}$	$\{632\}, \{53^2\}, \{62^21\}$
13	$\{6^21\}, \{651^2\}, \{5^221\}, \{542^2\}, \{4^232\}, \{43^3\}$	$\{643\}, \{63^21\}, \{54^2\}$
15	$\{6^221\}, \{652^2\}, \{5^232\}, \{543^2\}, \{4^33\}$	$\{654\}, \{64^21\}, \{5^3\}$
17	$\{6^232\}, \{653^2\}, \{5^243\}, \{54^3\}$	$\{6^25\}, \{65^21\}$
19	$\{6^243\}, \{654^2\}, \{5^34\}$	$\{6^31\}$
21	$\{65^3\}, \{6^254\}$	
23	$\{6^35\}$	
$[\tilde{\kappa}]$	$[1]$	$[5]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
1		
3	$\{1^3\}$	
5	$\{2^21\}, \{21^3\}$	
7	$\{3^21\}, \{321^2\}, \{2^31\}$	$\{51^2\}$
9	$\{4^21\}, \{431^2\}, \{3^221\}, \{32^3\}$	$\{52^2\}, \{61^3\}, \{521^2\}$
11	$\{5^21\}, \{541^2\}, \{4^221\}, \{432^2\}, \{3^32\}$	$\{53^2\}, \{62^21\}, \{5321\}$
13	$\{651^2\}, \{5^221\}, \{542^2\}, \{4^232\}, \{43^3\}$	$\{63^21\}, \{54^2\}, \{5431\}$
15	$\{652^2\}, \{5^232\}, \{543^2\}, \{4^33\}$	$\{64^21\}, \{5^3\}, \{5^241\}$
17	$\{653^2\}, \{5^243\}, \{54^3\}$	$\{65^21\}$
19	$\{654^2\}, \{5^34\}$	
21	$\{65^3\}$	
23		
$[\tilde{\kappa}]$	$[1^3]$	$[51^2]$

n	$\{\tilde{\lambda}\}$
3	$\{3\}$
5	$\{41\}, \{31^2\}$
7	$\{52\}, \{421\}, \{32^2\}, \{41^3\}$
9	$\{63\}, \{531\}, \{432\}, \{3^3\}, \{521^2\}, \{42^21\}$
11	$\{641\}, \{542\}, \{4^23\}, \{631^2\}, \{5321\}, \{43^21\}, \{52^3\}$
13	$\{652\}, \{6421\}, \{632^2\}, \{5^23\}, \{5431\}, \{53^22\}, \{4^31\}$
15	$\{6^23\}, \{6531\}, \{6432\}, \{63^3\}, \{5^241\}, \{54^22\}$
17	$\{6^241\}, \{6542\}, \{64^23\}, \{5^32\}$
19	$\{6^252\}, \{65^23\}$
21	$\{6^33\}$
$[\tilde{\kappa}]$	$[3]$

n	$\{\tilde{\lambda}\}$
3	$\{21\}$
5	$\{32\}, \{31^2\}, \{2^21\}, \{21^3\}$
7	$\{43\}, \{421\}, \{3^21\}, \{32^2\}, \{321^2\}^2, \{2^31\}$
9	$\{54\}, \{531\}, \{4^21\}, \{432\}, \{431^2\}^2, \{42^21\}, \{3^221\}^2, \{32^3\}$
11	$\{641\}, \{5^21\}, \{542\}, \{541^2\}^2, \{5321\}, \{4^221\}^2, \{43^21\}, \{432^2\}^2, \{3^32\}$
13	$\{652\}, \{651^2\}, \{6421\}, \{5^221\}^2, \{5431\}, \{542^2\}^2, \{53^22\}, \{4^232\}^2, \{43^3\}$
15	$\{6^221\}, \{6531\}, \{652^2\}, \{6432\}, \{5^232\}^2, \{54^22\}, \{543^2\}^2, \{4^33\}$
17	$\{6^232\}, \{6542\}, \{653^2\}, \{64^23\}, \{5^243\}^2, \{54^3\}$
19	$\{6^243\}, \{65^23\}, \{654^2\}, \{5^34\}$
21	$\{6^254\}$
$[\tilde{\kappa}]$	$[21]$

n	$\{\tilde{\lambda}\}$
5	$\{41\}$
7	$\{52\}, \{51^2\}, \{421\}, \{41^3\}$
9	$\{621\}, \{531\}, \{52^2\}, \{432\}, \{521^2\}^2, \{42^21\}$
11	$\{632\}, \{542\}, \{53^2\}, \{4^23\}, \{631^2\}, \{62^21\}, \{5321\}^2, \{43^21\}, \{52^3\}$
13	$\{643\}, \{6421\}, \{63^21\}, \{632^2\}, \{5^23\}, \{54^2\}, \{5431\}^2, \{53^22\}, \{4^31\}$
15	$\{654\}, \{6531\}, \{64^21\}, \{6432\}, \{5^241\}^2, \{54^22\}$
17	$\{6^241\}, \{65^21\}, \{6542\}, \{5^32\}$
19	$\{6^252\}$
$[\tilde{\kappa}]$	$[41]$

n	$\{\tilde{\lambda}\}$
5	$\{32\}$
7	$\{43\}, \{421\}, \{3^21\}, \{321^2\}$
9	$\{531\}, \{4^21\}, \{52^2\}, \{432\}, \{431^2\}^2, \{42^21\}, \{3^221\}, \{32^3\}$
11	$\{632\}, \{542\}, \{53^2\}, \{541^2\}, \{5321\}^2, \{4^221\}^2, \{43^21\}, \{432^2\}^2, \{3^32\}$
13	$\{643\}, \{6421\}, \{63^21\}, \{5^221\}, \{5431\}^2, \{542^2\}^2, \{53^22\}, \{4^232\}^2, \{43^3\}$
15	$\{6531\}, \{652^2\}, \{64^21\}, \{6432\}, \{5^232\}^2, \{54^22\}, \{543^2\}, \{4^33\}$
17	$\{6^232\}, \{6542\}, \{653^2\}, \{5^243\}$
19	$\{6^243\}$
$[\tilde{\kappa}]$	$[32]$

n	$\{\tilde{\lambda}\}$
5	$\{2^21\}$
7	$\{3^21\}, \{32^2\}, \{321^2\}, \{2^31\}$
9	$\{4^21\}, \{432\}, \{431^2\}, \{42^21\}, \{3^221\}^2, \{32^3\}$
11	$\{542\}, \{541^2\}, \{5321\}, \{4^221\}^2, \{43^21\}, \{432^2\}^2, \{3^32\}$
13	$\{6421\}, \{5^221\}, \{5431\}, \{542^2\}^2, \{53^22\}, \{4^232\}^2, \{43^3\}$
15	$\{652^2\}, \{6432\}, \{5^232\}, \{54^22\}, \{543^2\}^2, \{4^33\}$
17	$\{653^2\}, \{64^23\}, \{5^243\}, \{54^3\}$
19	$\{654^2\}$
$[\tilde{\kappa}]$	$[2^21]$

n	$\{\tilde{\lambda}\}$
5	
7	$\{421\}$
9	$\{52^2\}, \{432\}, \{521^2\}, \{431^2\}, \{42^21\}$
11	$\{53^2\}, \{4^23\}, \{62^21\}, \{5321\}^2, \{4^221\}, \{43^21\}, \{52^3\}, \{432^2\}$
13	$\{63^21\}, \{632^2\}, \{54^2\}, \{5431\}^2, \{542^2\}, \{53^22\}, \{4^31\}, \{4^232\}$
15	$\{64^21\}, \{6432\}, \{5^241\}, \{5^232\}, \{54^22\}$
17	$\{6542\}$
19	
$[\tilde{\kappa}]$	$[421]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
5	$\{21^3\}$	
7	$\{321^2\}, \{2^31\}$	$\{41^3\}$
9	$\{431^2\}, \{3^221\}, \{32^3\}$	$\{521^2\}, \{42^21\}$
11	$\{541^2\}, \{4^221\}, \{432^2\}, \{3^32\}$	$\{5321\}, \{43^21\}, \{52^3\}$
13	$\{5^221\}, \{542^2\}, \{4^232\}, \{43^3\}$	$\{5431\}, \{53^22\}, \{4^31\}$
15	$\{5^232\}, \{543^2\}, \{4^33\}$	$\{5^241\}, \{54^22\}$
17	$\{5^243\}, \{54^3\}$	$\{5^32\}$
19	$\{5^34\}$	
$[\tilde{\kappa}]$	$[21^3]$	$[41^3]$

n	$\{\tilde{\lambda}\}$
5	$\{31^2\}$
7	$\{421\}, \{32^2\}, \{41^3\}, \{321^2\}$
9	$\{531\}, \{432\}, \{3^3\}, \{521^2\}, \{431^2\}, \{42^21\}^2, \{3^221\}$
11	$\{542\}, \{4^23\}, \{631^2\}, \{541^2\}, \{5321\}^2, \{4^221\}, \{43^31\}^2, \{52^3\}, \{432^2\}$
13	$\{6421\}, \{632^2\}, \{5^23\}, \{5^221\}, \{5431\}^2, \{542^2\}, \{53^22\}^2, \{4^31\}, \{4^232\}$
15	$\{6531\}, \{6432\}, \{63^3\}, \{5^241\}, \{5^232\}, \{54^22\}^2, \{543^2\}$
17	$\{6542\}, \{64^23\}, \{5^32\}, \{5^243\}$
19	$\{65^23\}$
$[\tilde{\kappa}]$	$[31^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
7	$\{3^21\}$	$\{32^2\}$
9	$\{432\}, \{431^2\}, \{3^221\}$	$\{432\}, \{3^3\}, \{42^21\}, \{3^221\}$
11	$\{53^2\}, \{5321\}, \{4^221\}, \{43^21\}, \{432^2\}\{3^32\}$	$\{4^23\}, \{5321\}, \{4^221\}, \{43^21\}^2, \{52^3\}\{432^2\}$
13	$\{63^21\}, \{5431\}, \{542^2\}, \{53^22\}, \{4^232\}, \{43^3\}$	$\{632^2\}, \{5431\}, \{542^2\}, \{53^22\}^2, \{4^31\}, \{4^232\}$
15	$\{6432\}, \{5^232\}, \{543^2\}$	$\{6432\}, \{63^3\}, \{54^22\}, \{543^2\}$
17	$\{653^2\}$	$\{64^23\}$
$[\tilde{\kappa}]$	$[3^21]$	$[32^2]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
7	$\{321^2\}$	$\{2^31\}$	
9	$\{431^2\}, \{42^21\}, \{3^221\}, \{32^3\}$	$\{3^221\}, \{32^3\}$	$\{42^21\}$
11	$\{5321\}, \{4^221\}, \{43^21\}, \{432^2\}^2, \{3^32\}$	$\{4^221\}, \{432^2\}, \{3^32\}$	$\{43^21\}, \{52^3\}, \{432^2\}$
13	$\{5431\}, \{542^2\}, \{53^22\}, \{4^232\}^2, \{43^3\}$	$\{542^2\}, \{4^232\}, \{43^3\}$	$\{53^22\}, \{4^31\}, \{4^232\}$
15	$\{5^232\}, \{54^22\}, \{543^2\}, \{4^33\}$	$\{543^2\}, \{4^33\}$	$\{54^22\}$
17	$\{5^243\}$	$\{54^3\}$	
$[\tilde{\kappa}]$	$[321^2]$	$[2^31]$	$[42^21]$

n	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$	$\{\tilde{\lambda}\}$
9	$\{3^3\}$	$\{3^2 21\}$	$\{32^3\}$	
11	$\{43^2 1\}$	$\{43^2 1\}$, $\{432^2\}$, $\{3^3 2\}$	$\{432^2\}$, $\{3^3 2\}$	$\{3^3 2\}$
13	$\{53^2 2\}$	$\{53^2 2\}$, $\{4^2 32\}$, $\{43^3\}$	$\{4^2 32\}$, $\{43^3\}$	$\{43^3\}$
15	$\{63^3\}$	$\{543^2\}$	$\{4^3 3\}$	
$[\tilde{\kappa}]$	$[3^3]$	$[3^2 21]$	$[32^3]$	$[3^3 2]$

References

[1] Talmi I 1993 *Simple Models of Complex Nuclei* (Chur, Switzerland: Harwood Academic)
[2] Rowe D J, D'Agostino S, Carvalho M J and Repka J in preparation
[3] Racah G 1943 *Phys. Rev.* **63** 367
[4] Flowers B H 1952 *Proc. R. Soc.* A **212** 248
[5] Kerman A K 1961 *Ann. Phys., NY* **12** 300
[6] Helmers K 1961 *Nucl. Phys.* **23** 594
[7] Flowers B H and Spzikowski S 1964 *Proc. Phys. Soc.* **84** 193
[8] Flowers B H and Spzikowski S 1964 *Proc. Phys. Soc.* **84** 673
[9] Wigner E P 1937 *Phys. Rev.* **51** 106
[10] Bayman B F 1960 *Some Lectures on Groups and their Applications to Spectroscopy* (Copenhagen: Nordita)
[11] Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
[12] Chaćon E 1969 *PhD Thesis* University of México
[13] Kashiwara M and Vergne M 1978 *Invent. Math.* **44** 1
[14] Howe R 1979 *Proc. Symp. Pure Math.* **33** 275
Howe R 1985 *Lect. Appl. Math.* **21** 179
[15] Howe R 1995 Perspectives on invariant theory *The Schur Lectures (1992)* (*Israel Math. Conf. Proc.* **8** 1) (available from the American Mathematical Society, Providence, RI, USA)
[16] Evans J A, Dussel G G, Maqueda E E and Perazzo R P J 1981 *Nucl. Phys.* A **367** 77
[17] Dussel G G, Maqueda E E, Perazzo R P J and Evans J A 1986 *Nucl. Phys.* A **450** 164
[18] Bremond B 1968 *Nucl. Phys.* A **113** 257
[19] Pang S C 1969 *Nucl. Phys.* A **128** 497
[20] Engel J, Pittel S, Stoitsov M, Vogel P and Dukelsky J 1997 *Phys. Rev.* C **55** 1781
[21] Van Isacker P, Dukelsky J, Pittel S and Juillet O 1998 *J. Phys. G: Nucl. Part. Phys.* **24** 1261
[22] Kota V K B and Castilho Alcarás J A 2006 *Nucl. Phys.* A **764** 181
[23] Murnaghan F D 1938 *Theory of Group Representations* (Baltimore, MD: John Hopkins University Press)
[24] Littlewood D E 1950 *The Theory of Group Characters* (Oxford: Oxford University Press)
[25] Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A: Math. Gen.* **18** 939
[26] King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
[27] Elliott J P 1958 *Proc. R. Soc.* A **245** 128, 562